Basic Tools of Control Theory

Outline

- Open- and Closed-Loop Control
- Laplace Transform
- The Canonical Spring-Mass-Damper
- Equilibrium Setpoint Control
- Qualitative Second-Order Response
- Closed-Loop Transfer Function
- Time- and Frequency-Domain Response

Open- and Closed-Loop Control

open-loop -
an initiating stimulus causes a response without further stimulation

- SR reflexes - e.g. withdrawl reflex
- fire alarm

closed-loop -
a (time-varying) setpoint is achieved by constantly measuring and correcting in order to actively reject disturbances

- autopilot
- Norbert Weiner - cybernetics (helmsman), homeostasis, endocrine system living things as feedback regulators

Related Segmental and Intersegmental Reflexes

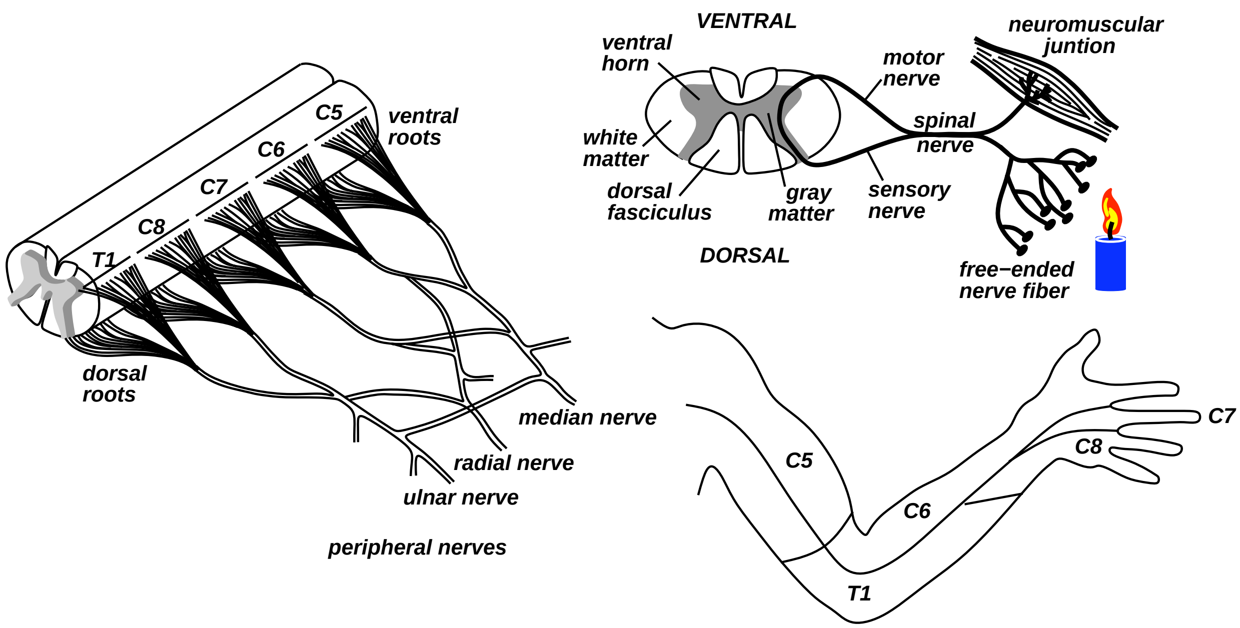
withdrawl reflex movement is initiated by the free-ended nerve fiber in response to painful stimuli rather than the muscle spindle. The number of body parts enlisted by the reflex is proportional to the strength of the painful stimulus. Two synapses are involved in this reflex that can be inhibited cortically via an interneuron.

Descarte - the *animal spirit* expressed in the form of a reflex

intersegmental extensor reflex the flexor (withdrawl) reflex on one arm, for instance is accompanied about 0.5 *seconds* later by a contralateral extensor response.
reciprocal inhibition keeps antagonistic pairs from fighting each other. It is especially important in walking as contact with the ground causes reaction loads.

The Central Nervous System - Spinal Processing

spinal cord: about 1 *cm* in diameter and protected within the bony vertebral column - cervical, thoracic, lumbar, sacral, and coccygeal.



gray matter spinal sensory and motor cells and **white matter** ascending and descending tracts - gets its color from neuroglial cells that act as insulators for the ascending and descending pathways.

thirty one pairs of spinal nerves

Laplace Transform

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

where $s = \sigma + j\omega$

The Laplace integral will converge if:

- $f(t)$ is piecewise continuous,
- $f(t)$ is of exponential order — i.e., there exists an a such that $|f(t)| \leq Me^{at}$ for all $t > T$ where T is some finite time.

Name	Theorem
Derivative	$\mathcal{L}[\frac{df}{dt}] = sF(s) - f(0+)$
Integral	$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$
Shifting	$\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-t_0 s}F(s)$

...a linear differential equation with constant coefficients and a finite number of terms is Laplace-transformable...they transform into polynomials in "s."

Laplace Transform

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

transforms an N^{th} order differential equation

$$a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0 = 0$$

into an N^{th} order polynomial

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$$

CHARACTERISTIC EQUATION

roots of the characteristic equation
determine the form of the response

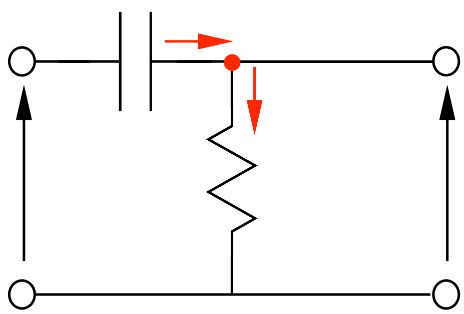
Laplace Transform Pairs

Name	f(t)	F(s)
unit impulse	$\delta(t)$	1
unit step	$u(t)$	$\frac{1}{s}$
ramp	t	$\frac{1}{s^2}$
n^{th}-order ramp	t^n	$\frac{n!}{s^{n+1}}$
exponential	e^{-at}	$\frac{1}{s+a}$
ramped exponential	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$
sine	$\sin at$	$\frac{a}{s^2+a^2}$
cosine	$\cos at$	$\frac{s}{s^2+a^2}$
damped sine	$e^{-at}sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
damped cosine	$e^{-at}sin\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
hyperbolic sine	$\sinh at$	$\frac{a}{s^2-a^2}$
hyperbolic cosine	$\cosh at$	$\frac{s}{s^2-a^2}$

Transfer Functions

...describe dynamic responses in a simple linear model...

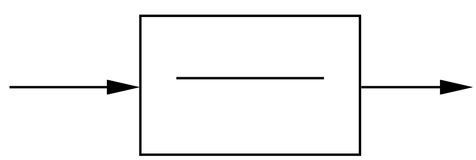
Example: and RC Circuit



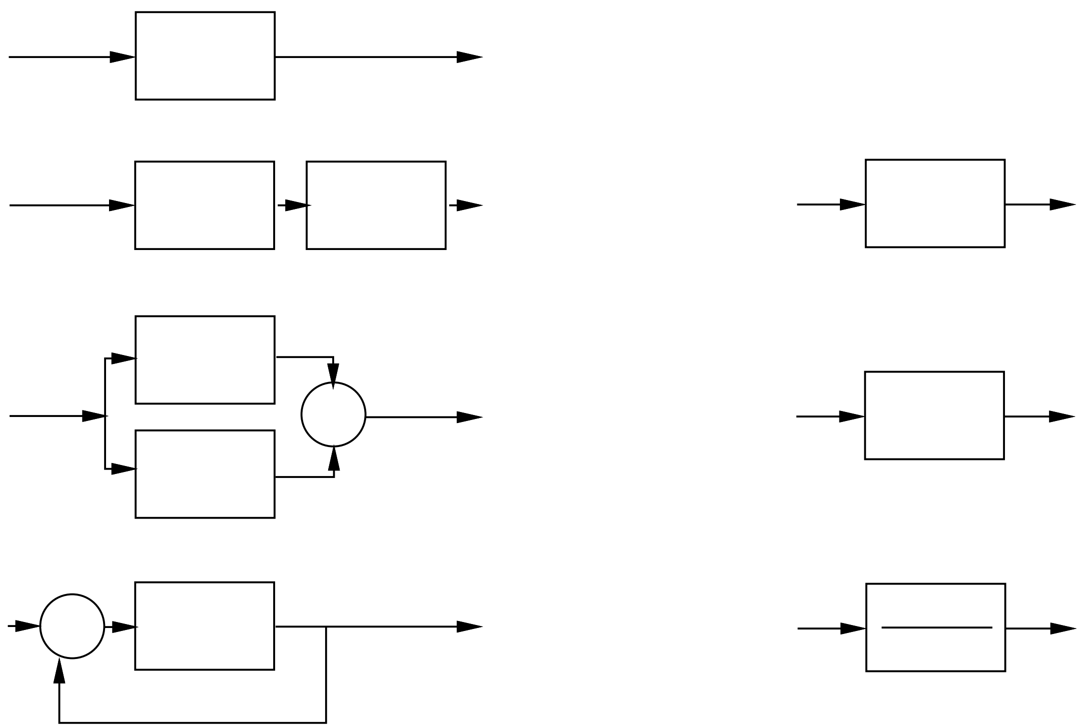
$$V_{out}(t) = i(t)R$$
$$V_{in}(t) = i(t)R + \frac{1}{C}\int i(t)dt$$

$$V_{out}(s) = I(s)R$$
$$V_{in}(s) = I(s)(R + \frac{1}{Cs})$$

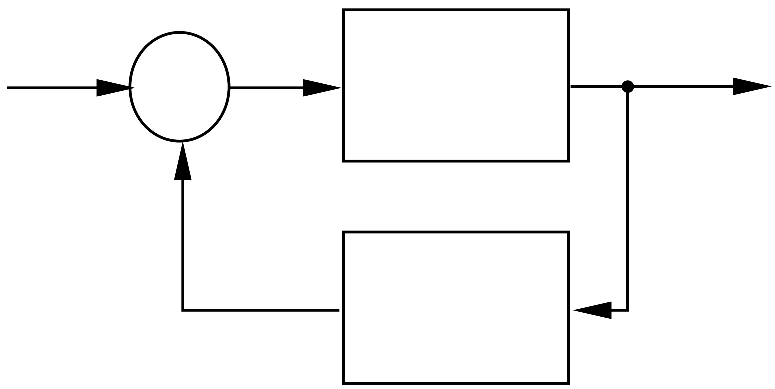
$$G = \frac{V_{out}}{V_{in}} = \frac{RCs}{1+RCs}$$



Transfer Functions



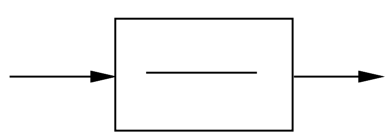
Closed-Loop Transfer Function



$$
\begin{aligned}
C &= G\,E \\
E &= R - B, \quad \text{and} \\
B &= H\,C.
\end{aligned}
\qquad
\frac{C}{R} = \frac{GE}{R} = \frac{G(R-B)}{R} = \frac{G(R-HC)}{R}
$$

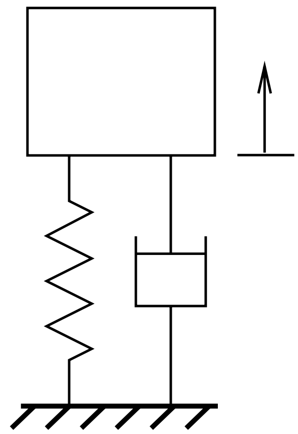
$$
\begin{aligned}
\frac{C}{R} + \frac{GHC}{R} &= \frac{GR}{R} \\
\frac{C(1+GH)}{R} &= G, \qquad \text{and}
\end{aligned}
$$

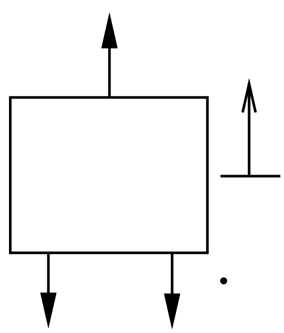
$$\frac{C}{R} = \frac{G}{1+GH}$$



The Spring-Mass-Damper

$$F_b = -Bv = -B\dot{x},$$
$$F_k = -Kx$$





$$\sum F = M\ddot{x} = f(t) - B\dot{x} - Kx$$

$$M\ddot{x} + B\dot{x} + Kx = f(t), \qquad \text{or}$$

$$\ddot{x} + (B/M)\dot{x} + (K/M)x = f(t)/M$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)/M,$$

where

$$\omega_n = (K/M)^{1/2} \qquad [rad/sec] \text{ - } \textit{natural frequency}$$
$$\zeta = B/2(KM)^{1/2} \qquad 0 \leq \zeta \leq \infty \text{ - } \textit{damping ratio}$$

Equilibrium Setpoint Control

$$f(t)/M = \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x$$

$$F(s) = \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right) X(s), \qquad \text{so that,}$$

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

steady state displacement - open-loop position controller

$$F(S) = F_{const} = KX(s)_{ref}$$

$$\begin{aligned} KX_{ref}(s) &= \left(Ms^2 + Bs + K\right) X_{act}(s), \qquad \text{so that,} \\ \frac{X_{act}(s)}{X_{ref}(s)} &= \frac{K}{Ms^2 + Bs + K} \end{aligned}$$

$$\frac{X_{act}(s)}{X_{ref}(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

..input position reference, $X_{ref}(s) = F(s)/K$ achieves $X_{ref}(s)$ asymptotically as $t \to \infty$...

Qualitative Second-Order Response

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

is quadratic in s and has, therefore, two roots in general. The form of the response in the time domain, therefore, $x(t) = A_0 + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

case(1): $\zeta < 1$ (underdamped) - roots s_1 and s_2 are complex conjugates resulting in an oscillatory response ($e^{i\omega t} = cos(\omega t) + isin(\omega t)$).

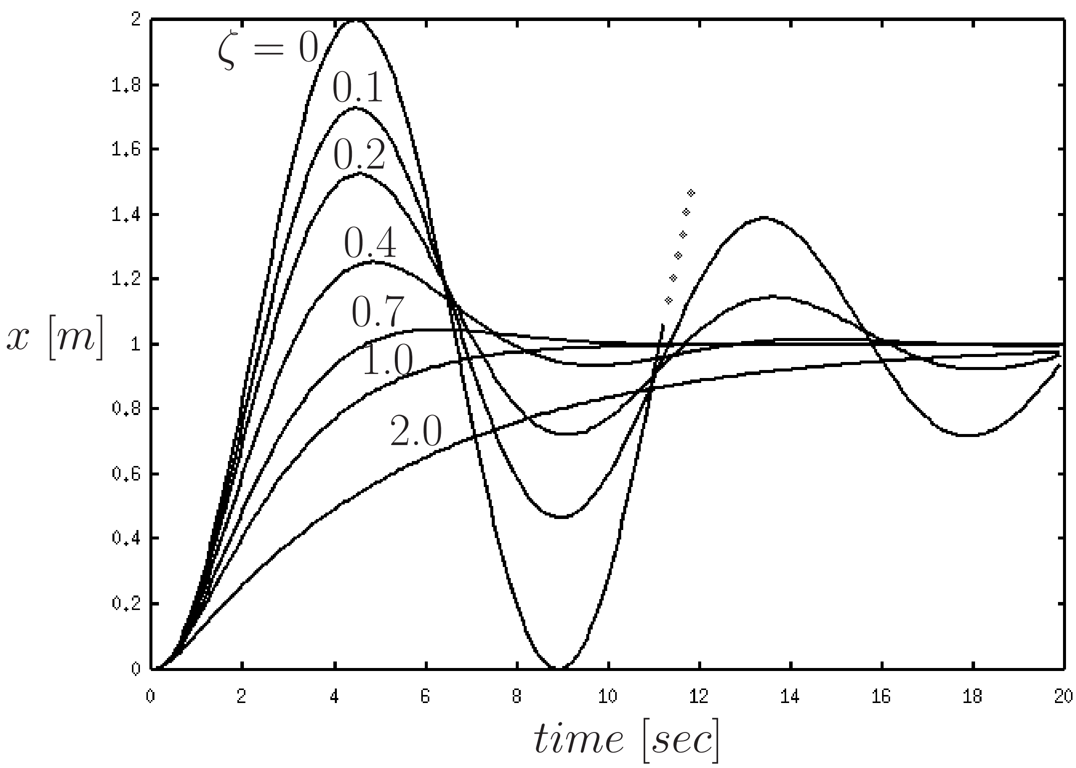
case(2): $\zeta > 1$ (overdamped) - s_1 and s_2 are distinct real roots, the asymptotic response is dominated by the root with the smallest absolute value.

case(3): $\zeta = 1$ (critically damped) - repeated real roots.

$$x(t) = A_0 + A_1 e^{st} + A_2 t e^{st}$$

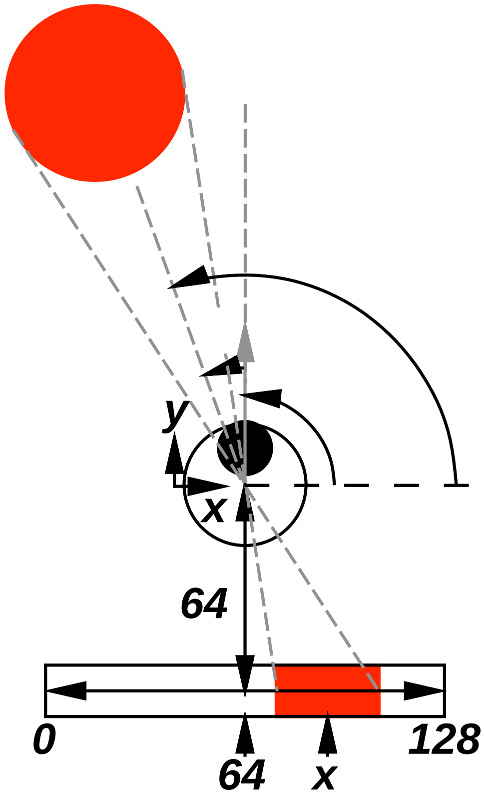
Qualitative Second-Order Response - continued

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$



$(K = 1.0\ [N/m],\ M = 2.0\ [kg])$

Ocularmotor Pursuit in Roger



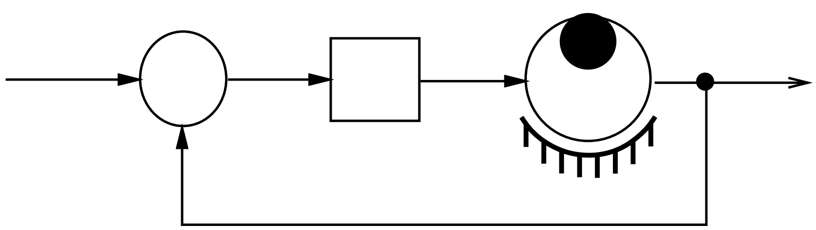
We wish to keep the center of the red ball in the center of the field of view.

...the antagonistic behavior of the lateral rectus and medial rectus muscles in the human eye...

$$\tau_{motor} = \alpha(\theta_{ref} - \theta_{act}) = \alpha e(t)$$

where α is the amplifier gain.

$$\sum \tau = J\ddot{\theta}_{act} = \tau_{motor} - B\dot{\theta}_{act}$$



$$(Js^2 + Bs)\Theta_{act}(s) = \tau(s) = \alpha E(s)$$

Ocularmotor Pursuit in Roger - continued

The Closed-Loop Transfer Equation

feedforward transfer function, G:

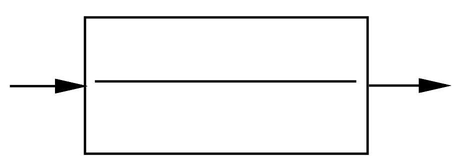
$$G = \frac{\Theta_{act}(s)}{E(s)} = \frac{\alpha}{Js^2 + Bs}$$

$$\frac{C}{R} = \frac{\Theta_{act}(s)}{\Theta_{ref}(s)} = \frac{G}{1 + GH} = \frac{\alpha/(Js^2 + Bs)}{1 + \alpha/(Js^2 + Bs)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

where:

$$\begin{aligned} \omega_n &= \sqrt{\alpha/J} \\ \zeta &= B/2(\alpha J)^{1/2} \end{aligned}$$

This closed-loop transfer function accomplishes the *same* second order transformation as the SMD



Ocularmotor Pursuit in Roger - continued

The Time Domain Response

...at $t = 0$, apply a unit step reference input

$$r(t) = 1$$
$$R(s) = \frac{1}{s}$$

Therefore, if we let $\omega_n = 1$ and $\zeta = 1$

$$\Theta_{act}(s) = \frac{1}{s(s+2)+1}\frac{1}{s} = \frac{1}{s(s+1)^2}$$

partial-fraction expansion of this quotient yields:

$$\Theta_{act}(s) = \frac{1}{s(s+1)^2} = \frac{a}{s} + \frac{b}{(s+1)} + \frac{c}{(s+1)^2}$$
$$= \frac{1}{s} + \frac{-1}{(s+1)} + \frac{-1}{(s+1)^2}$$

The inverse Laplace transform (from the tables)

$$\theta_{act}(t) = 1 - e^{-t} - te^{-t}$$

so that at $t = 0$, $\theta_{act}(t) = 0$, but as $t \rightarrow \infty$, the robot converges to the reference position.

Frequency-Domain Response

consider a sinusoidal input with frequency ω.

$$\begin{aligned} r(t) &= A\cos\omega t \\ R(s) &= \frac{As}{s^2+\omega^2} \end{aligned}$$

the output is the product of the closed-loop transfer function and the input,

$$C(S) = \frac{G}{1+GH}\,\frac{As}{s^2+\omega^2} = \frac{G}{1+GH}\,\frac{As}{(s-i\omega)(s+i\omega)}$$

The partial fraction expansion of the quotient yields

$$C(S) = C_{cltf}(s) + \frac{k_1}{s-i\omega} + \frac{k_2}{s+i\omega}.$$

the last two terms introduce roots at $s = \pm i\omega$ and the inverse Laplace transform of these terms yields time domain responses like:

$$k_1 e^{i\omega t} \text{ and, } k_2 e^{-i\omega t}$$

...the steady state response of the second order system in response to a sinusoidal input is also a sinusoid of the same frequency...

Frequency-Domain Response - continued

the magnitude of the sinusoidal response will be proportional to the amplitude of the forcing function, A, and the gain expressed in the closed-loop transfer function,

$$\frac{G(s)}{1+G(s)H(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{(s/\omega_n)^2+2\zeta(s/\omega_n)+1}$$

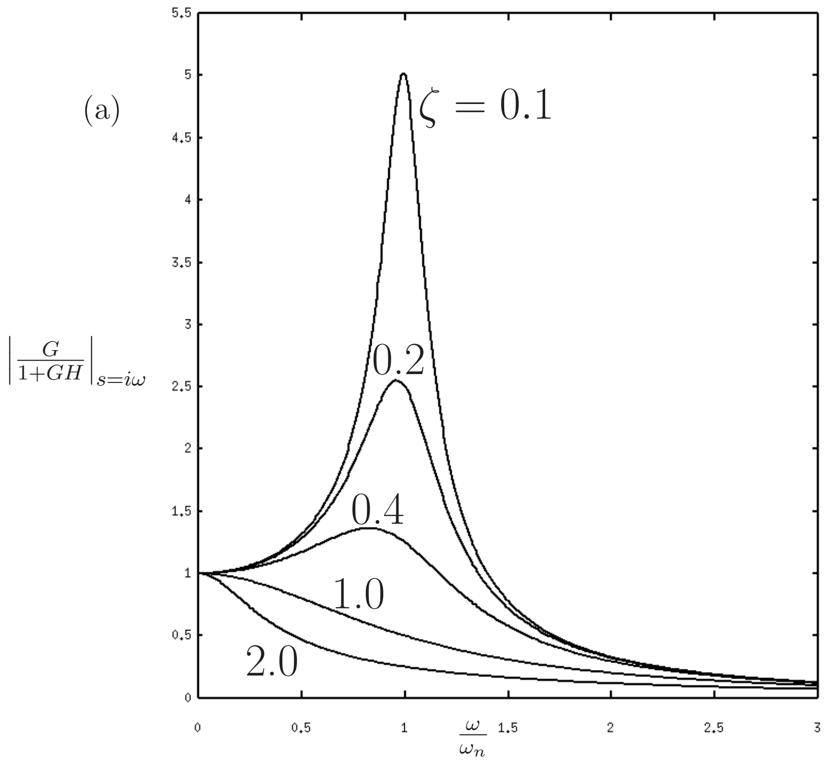
The gain from the CLTF can be determined by evaluating the CLTF at the roots introduced by the forcing function ($s = \pm i\omega$). The result is a complex number with corresponding magnitude and phase:

$$\left|\frac{G(s)}{1+G(s)H(s)}\right|_{s=i\omega} = \frac{1}{\left[(1-(\omega/\omega_n)^2)^2+(2\zeta(\omega/\omega_n))^2\right]^{1/2}}$$

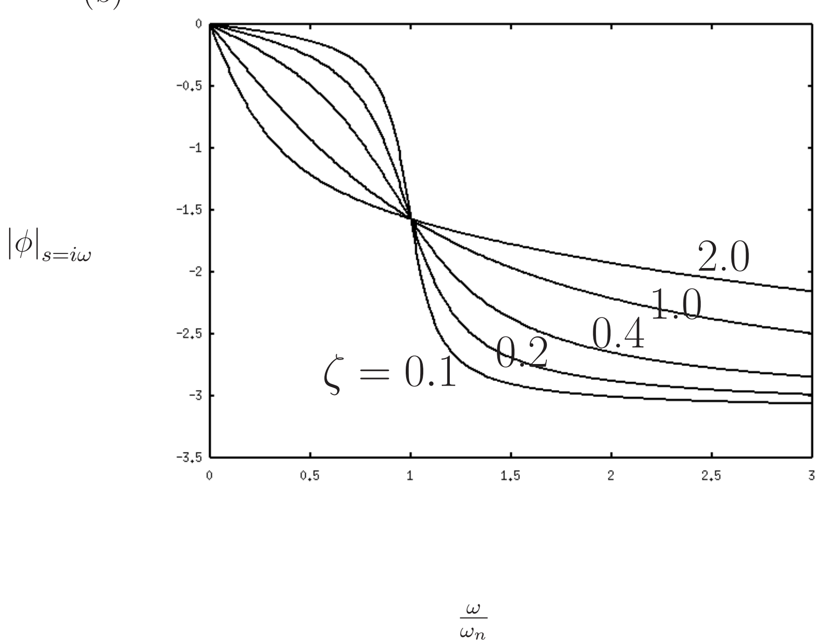
$$\phi(\omega) = -tan^{-1}\left(\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right)$$

Frequency-Domain Response

(a)



(b)



Frequency-Domain Response

- for $\zeta = 0$, the gain becomes theoretically infinite.
- for large driving frequencies, the gain in the CLTF goes to zero asymptotically
- the *bandwidth* of the system is that frequency where the gain falls to $1/\sqrt{2}$ of the DC response.
- the natural frequency identifies the point at which the response lags 90 degrees behind the reference input
- for driving frequencies greater than the natural frequency, the response goes asymptotically toward 180 degrees out of phase with the forcing function.

What is this?