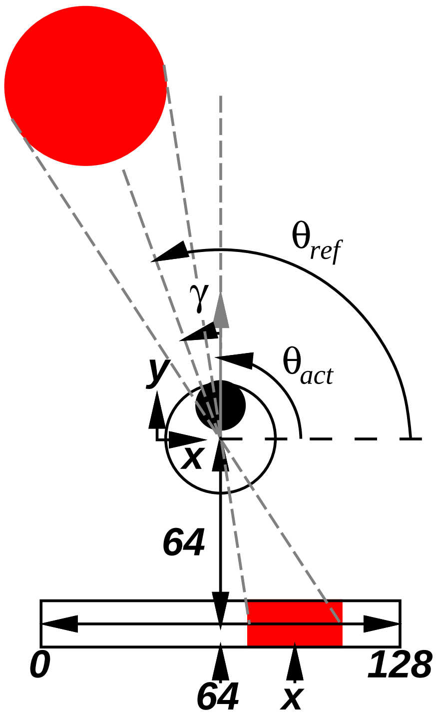
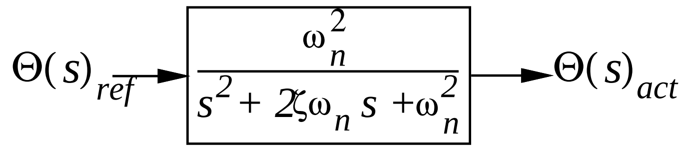
Time- and Energy-Based Stability

Outline

- Root Locus
- Lypunov's methods

Root Locus





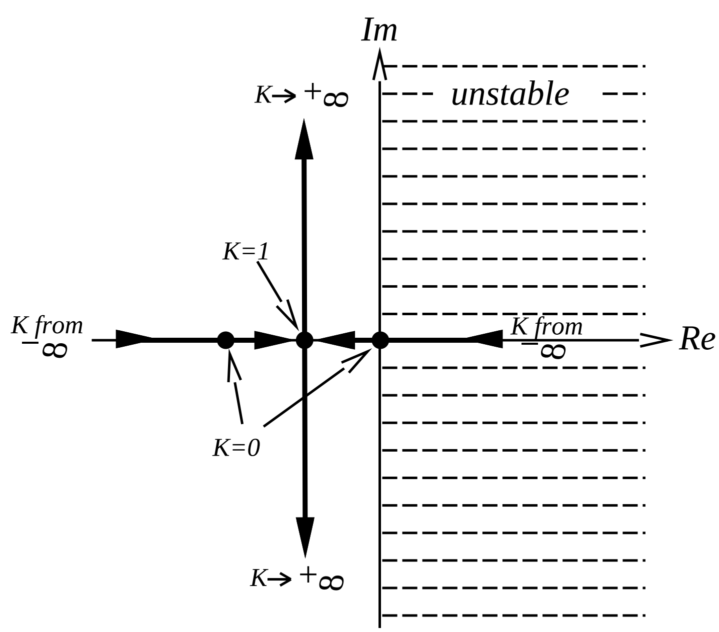
The homogeneous (unforced) differential equation of motion:

$$1 + GH = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

stability: *bounded input* → *bounded output* (BIBO)

consider:

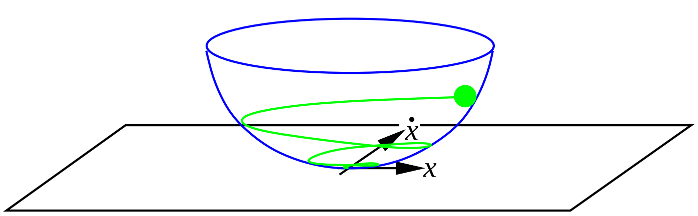
$$1 + GH = s^2 + 2s + K$$

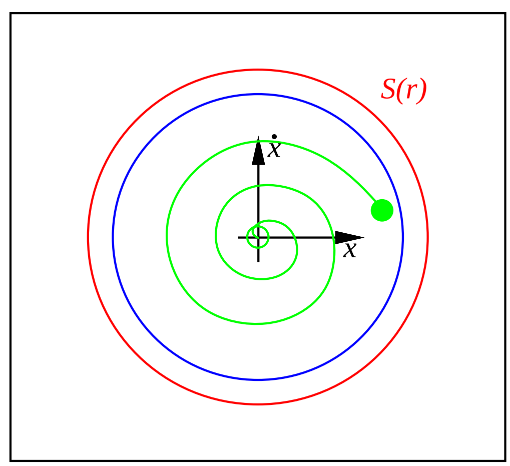


Analytic Stability — Lyapunov's Direct Method

Stability - the origin of the state space is stable if there exists a region, $S(r)$, such that states which start within $S(r)$ remain within $S(r)$.

Asymptotic Stability - a system is asymptotically stable in $S(r)$ if as $t \rightarrow \infty$, the system state approaches the origin of the state space.





Analytic Stability - Lyapunov's Second Method

Define: an arbitrary scalar function, $V(\mathbf{x}, t)$, called a *Lyapunov function*, continuous is all first derivatives, where $\mathbf{x}$ is the state and t is time,

Iff: If the function, $V(\mathbf{x}, t)$, exists such that:

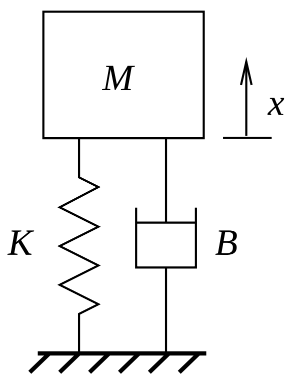
$$\begin{array}{llll} (a)\ V(0,t) & = & 0,\ and & \\ (b)\ V(\mathbf{x},t) & > & 0,\ for\ x \neq 0 & (positive\ definite),\ and \\ (c)\ \partial V/\partial t & < & 0 & (negative\ definite), \end{array}$$

Then: the system described by V is asymptotically stable in the neighborhood of the origin.

...if a system is stable, then there exists a suitable Lyapunov function.

...if, however, a particular Lyapunov function does not satisfy these criteria, it is not necessarily true that this system is unstable.

EXAMPLE: spring-mass-damper



system dynamics:

$$\ddot{x} + \frac{B}{M}\dot{x} + \frac{K}{M}x = 0$$

$$\begin{aligned} E &= \int_0^v (Mv)dv + \int_0^x (Kx)dx \\ &= \frac{1}{2}Mv^2 + \frac{1}{2}Kx^2 \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Kx^2 \end{aligned}$$

Lyapunov function:

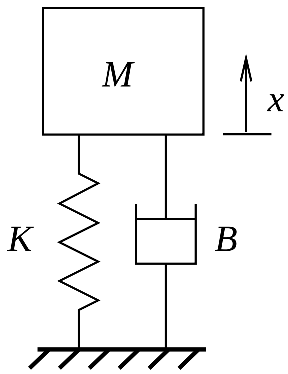
$$V(\mathbf{x}, t) = E = \frac{M\dot{x}^2}{2} + \frac{Kx^2}{2}$$

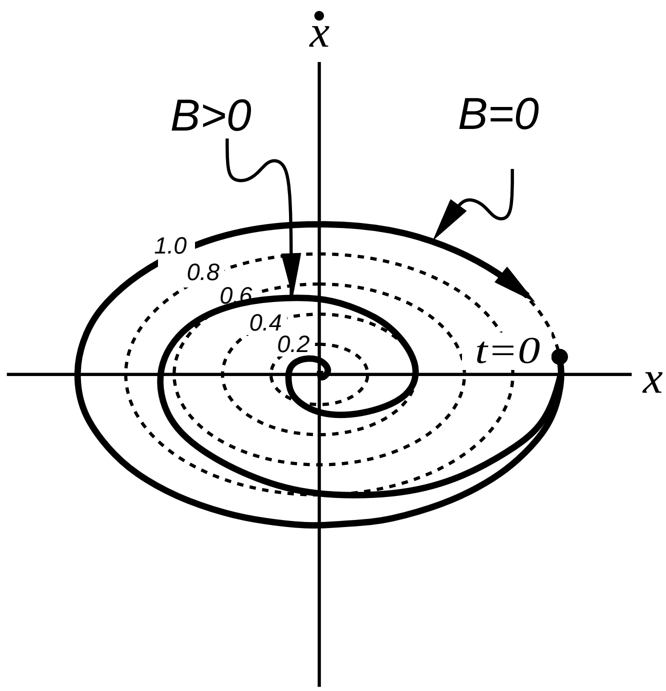
$(a)\ V(0, t) \ = \ 0, \qquad \surd$

$(b)\ V(\mathbf{x}, t) \ > \ 0, \qquad \surd$

$(c)\ \partial V/\partial t \qquad$ *negative definite?*

EXAMPLE: spring-mass-damper





...the entire state space is asymptotically stable for $B > 0$.

EXAMPLE: population dynamics

system dynamics:

$x_1 = \#$ males $\qquad \dot{x}_1 = -x_1 + \alpha x_1 x_2 = x_1(\alpha x_2 - 1)$

$x_2 = \#$ females $\qquad \dot{x}_2 = -x_2 + \beta x_1 x_2 = x_2(\beta x_1 - 1)$

equilibrium points: $\dot{\mathbf{x}} = 0$

(a) $x_1 = x_2 = 0$
(b) $x_1 = \frac{1}{\beta}, x_2 = \frac{1}{\alpha}$

Lyapunov function:

$$\left.\begin{array}{l} V(0,t) = 0 \\ V(\mathbf{x},t) > 0 \end{array}\right\} \quad \text{choose } V(\mathbf{x},t) = x_1^2 + x_2^2$$

$$\begin{aligned} \frac{\partial V}{\partial t} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1^2(\alpha x_2 - 1) + 2x_2^2(\beta x_1 - 1) \leq 0 \end{aligned}$$

EXAMPLE: population dynamics