Dynamics

The branch of physics that treats the action of force on bodies in motion or at rest; kinetics, kinematics, and statics, collectively. — *Websters dictionary*

Outline

- Conservation of Momentum
- Inertia Tensors - translation and rotation
- Dynamics
  - Newton/Euler Dynamics
  - State Space Form - computed torque equation
- Applications:
  - simulation;
  - control - feedforward compensation;
  - analysis - the acceleration ellipsoid.

Newton's Laws

1. a particle will remain in a state of constant rectilinear motion unless acted on by an external force;

2. the time-rate-of-change in the momentum (mv) of the particle is proportional to the externally applied forces, $F = \frac{d}{dt}(mv)$; and

3. any force imposed on body A by body B is reciprocated by an equal and opposite reaction force on body B by body A.

Conservation of Momentum

Linear:

$$F = \frac{d}{dt}[m\dot{x}] = m\ddot{x}$$

$$[\mathrm{N}] = \left[\frac{\mathrm{kg\ m}}{\mathrm{sec}^2}\right]$$

Angular:

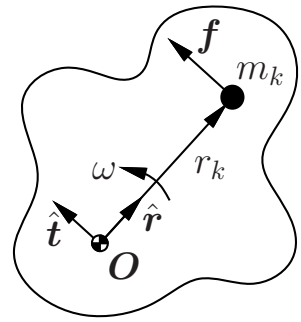
$$\tau = \frac{d}{dt}\left[J\dot{\theta}\right] = J\ddot{\theta}$$

$$[\mathrm{Nm}] = \left[\frac{\mathrm{kg\ m}^2}{\mathrm{sec}^2}\right]$$

J is called the mass moment of inertia

Conservation of Momentum

To generate an angular acceleration about $\boldsymbol{O}$, a torque is applied around the $\hat{\boldsymbol{z}}$ axis



$$\begin{aligned}\boldsymbol{\tau}_k &= \boldsymbol{r} \times \boldsymbol{f} = r_k \hat{\boldsymbol{r}} \times \frac{d}{dt}(m_k \boldsymbol{v}_k) \\ &= m_k r_k \left[\hat{\boldsymbol{r}} \times \frac{d}{dt}(\boldsymbol{v}_k) \right]\end{aligned}$$

the velocity of m_k due to ω_O is

$$\boldsymbol{v}_k = (\omega \hat{\boldsymbol{z}} \times r_k \hat{\boldsymbol{r}}) = (r_k \omega) \hat{\boldsymbol{t}},$$

so that

$$\boldsymbol{\tau}_k = (m_k r_k^2) \dot{\omega} \; \hat{\boldsymbol{z}} = J_k \dot{\omega} \; \hat{\boldsymbol{z}}$$

where, J_k is the mass moment of inertia of particle m_k around the $\hat{\boldsymbol{z}}$ axis located at frame $\boldsymbol{O}$

$$\boldsymbol{\tau} = \left(\sum_k m_k r_k^2 \right) \dot{\omega} = J \dot{\omega}.$$

Conservation of Momentum

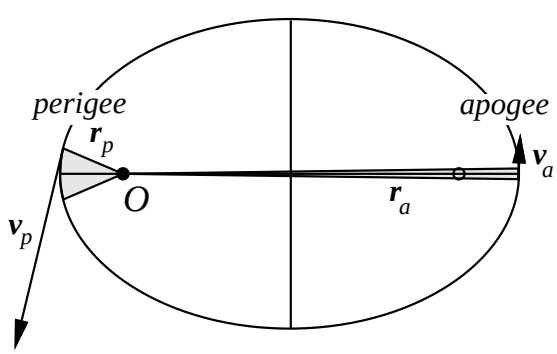
J $[kg \cdot m^2]$ is the scalar *moment of inertia*

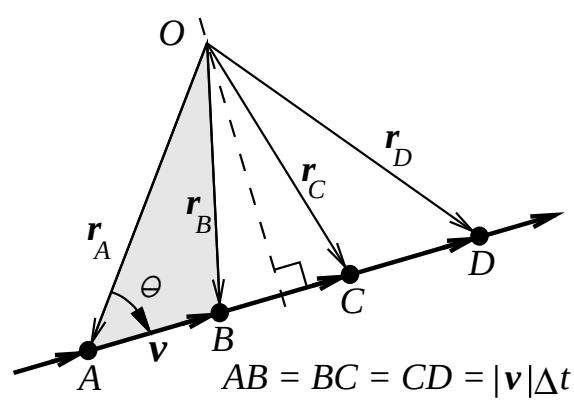
In the rotating lamina, the counterpart of linear momentum ($p = m\boldsymbol{v}$) is *angular momentum* $L = J\boldsymbol{\omega}$,

so that Euler's equation can be written in the same form as Newton's second law

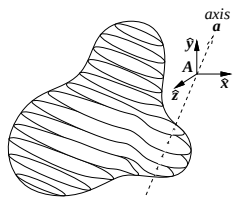
$$\boldsymbol{\tau} = \frac{d}{dt}\left[J\omega\right] = J\ddot{\theta}$$

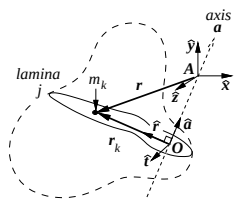
...rotating bodies conserve angular momentum (and remain in a constant state of angular velocity) unless acted upon by an external torque...





Inertia Tensor

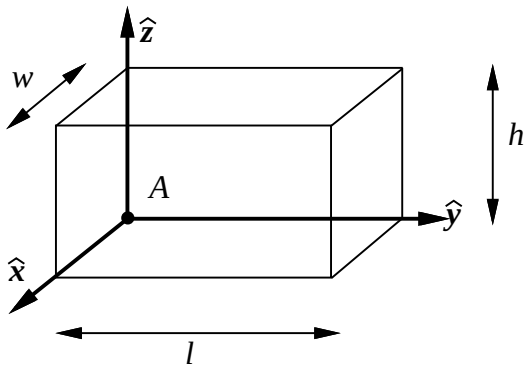




$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

MASS MOMENTS OF INERTIA	MASS PRODUCTS OF INERTIA
$I_{xx} = \int\int\int (y^2 + z^2)\rho dv$	$I_{xy} = \int\int\int xy\rho dv$
$I_{yy} = \int\int\int (x^2 + z^2)\rho dv$	$I_{xz} = \int\int\int xz\rho dv$
$I_{zz} = \int\int\int (x^2 + y^2)\rho dv$	$I_{yz} = \int\int\int yz\rho dv$

EXAMPLE: Inertia Tensor of an Eccentric Rectangular Prism



$$
\begin{aligned}
I_{xx} &= \int_0^h \int_0^l \int_0^w (y^2 + z^2)\rho dx dy dz \\
&= \int_0^h \int_0^l (y^2 + z^2) w\rho dy dz \\
&= \int_0^h \left[(\frac{y^3}{3} + z^2 y) \right]_0^l w\rho dz \\
&= \int_0^h \left(\frac{l^3}{3} + z^2 l \right) w\rho dz \\
&= \left(\frac{l^3 z}{3} + \frac{l z^3}{3} \right) \Bigg|_0^h (w\rho) \\
&= \left(\frac{l^3 h}{3} + \frac{l h^3}{3} \right) w\rho
\end{aligned}
$$

or, since the mass of the rectangle $m = (wlh)\rho$,

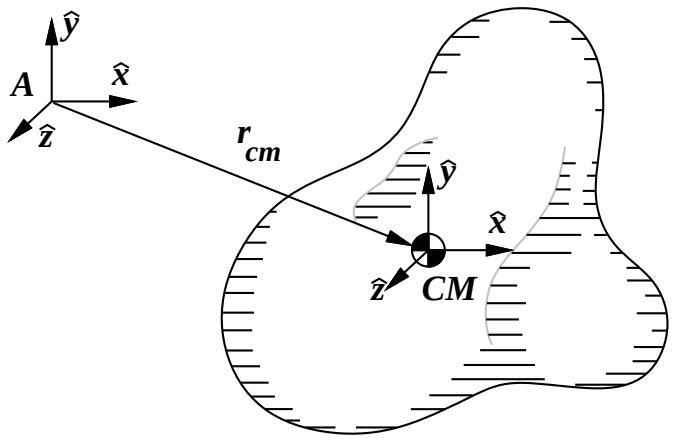
$$I_{xx} = \frac{m}{3}(l^2 + h^2).$$

EXAMPLE: Inertia Tensor of an Eccentric Rectangular Prism

...completing the other moments and products of inertia yields:

$$
{}^{A}I = \begin{bmatrix} \frac{m}{3}(l^2 + h^2) & \frac{m}{4}wl & \frac{m}{4}hw \\ \frac{m}{4}wl & \frac{m}{3}(w^2 + h^2) & \frac{m}{4}hl \\ \frac{m}{4}hw & \frac{m}{4}hl & \frac{m}{3}(l^2 + w^2) \end{bmatrix}
$$

Parallel Axis Theorem - Translating the Inertia Tensor



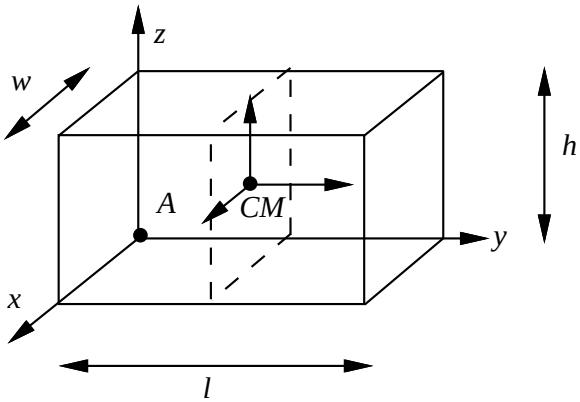
the moments of inertia look like:

$$^{A}I_{zz} = {}^{CM}I_{zz} + m(r_x^2 + r_y^2),$$

and the products of inertia are:

$$^{A}I_{xy} = {}^{CM}I_{xy} + m(r_x\ r_y).$$

EXAMPLE:
The Symmetric Rectangular Prism



$$
\begin{aligned}
{}^{CM}I_{zz} &= {}^{A}I_{zz} - m(r_x^2 + r_y^2) \\
&= \frac{m}{3}(l^2 + w^2) - \frac{m}{4}(l^2 + w^2) \\
&= \frac{m}{12}(l^2 + w^2)
\end{aligned}
$$

$$
\begin{aligned}
{}^{CM}I_{xy} &= {}^{A}I_{xy} - m(r_x\, r_y) \\
&= \frac{m}{4}(wl) - \frac{m}{4}(wl) = 0.
\end{aligned}
$$

moving the axes of rotation to the center of mass results in a diagonalized inertia tensor

$$
{}^{CM}I = \frac{m}{12}\begin{bmatrix} (l^2 + h^2) & 0 & 0 \\ 0 & (w^2 + h^2) & 0 \\ 0 & 0 & (l^2 + w^2) \end{bmatrix}
$$

diagonal terms are smaller and the off-diagonals are 0

Rotating the Inertia Tensor

angular momentum $\mathbf{L}_0 = \mathbf{I}_0\boldsymbol{\omega}$ about frame 0 in a vector quantity that is conserved.

we can express it relative to frame 1 as

$$\mathbf{L}_1 = {}_1R_0\mathbf{L}_0$$

or

$$\begin{aligned}\mathbf{I}_1\boldsymbol{\omega}_1 &= {}_1R_0(\mathbf{I}_0\boldsymbol{\omega}_0) \\ &= {}_1R_0\mathbf{I}_0[\,{}_1R_0^T\,{}_1R_0\,]\boldsymbol{\omega}_0 = {}_1R_0\,\mathbf{I}_0\,{}_1R_0^T\boldsymbol{\omega}_1\end{aligned}$$

and therefore,

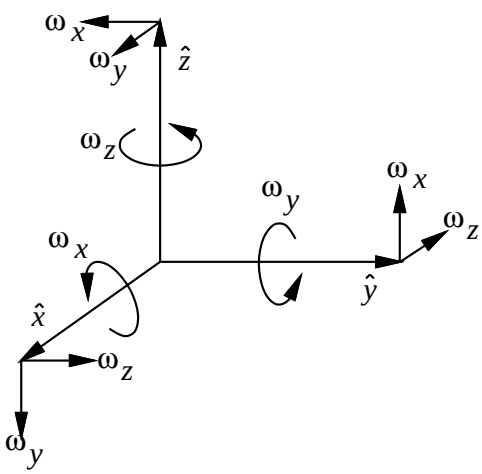
$$\mathbf{I}_1 = {}_1R_0\,\mathbf{I}_0\,{}_1R_0^T.$$

Rotating Coordinate Systems

Definition (Inertial Frame)
the frame where the absolute state of motion is completely known

Let frame A be an inertial frame. Frame B has an absolute velocity, $\boldsymbol{\omega}_B^B$ (written in terms of frame B coordinates).

$$\begin{aligned}
\mathbf{r}_A(t) &= {}_AR_B(t)\mathbf{r}_B(t) \\
\dot{\mathbf{r}}_A(t) &= {}_AR_B(t)\frac{d}{dt}[\mathbf{r}_B(t)] + \frac{d}{dt}[\ {}_AR_B(t)\ ]\mathbf{r}_B(t)
\end{aligned}$$



To evaluate the second term on the right, consider how the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, basis vectors for frame B change by virtue of $\boldsymbol{\omega}_B$.

$$\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \qquad\quad +\omega_z\hat{\boldsymbol{y}} \ -\omega_y\hat{\boldsymbol{z}} \\
\dot{\hat{\boldsymbol{y}}} &= -\omega_z\hat{\boldsymbol{x}} \qquad\quad +\omega_x\hat{\boldsymbol{z}} \\
\dot{\hat{\boldsymbol{z}}} &= +\omega_y\hat{\boldsymbol{x}} \ -\omega_x\hat{\boldsymbol{y}}
\end{aligned}$$

SO

$$\begin{aligned}
\frac{d}{dt}[\ {}_AR_B(t)\ ]\mathbf{r}_B(t) &= \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \\
&= \boldsymbol{\omega} \times \mathbf{r}
\end{aligned}$$

Rotating Coordinate Systems

Therefore,

$$
\begin{aligned}
\dot{\mathbf{r}}_A(t) &= {}_AR_B(t)\frac{d}{dt}[\mathbf{r}_B(t)] + \frac{d}{dt}[\ {}_AR_B(t)\ ]\mathbf{r}_B(t) \\
&= {}_AR_B\left[\dot{\mathbf{r}}_B + (\boldsymbol{\omega}_B^B \times \mathbf{r}_B)\right]
\end{aligned}
$$

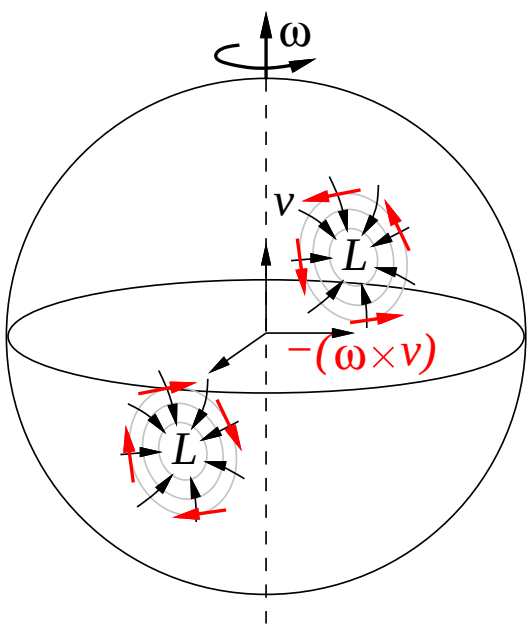
and, in fact, all vector quantities expressed in local frames that are moving relative to an inertial frame are differentiated in this way

$$
\frac{d}{dt}\left[{}_A\mathbf{R}_B(t)(\cdot)_B\right] = {}_A\mathbf{R}_B\left[\frac{d}{dt}(\cdot)_B + (\omega_B \times (\cdot)_B)\right]
$$

gives rise to the notorious Coriollis and centripetal forces!

Rotating Coordinate Systems: Low Pressure Systems

large scale atmospheric flows converge at low pressure regions. A nonrotating planet, this would result in flow lines directed radially inward.



but the earth rotates...
consider a stationary inertial frame A and a rotating frame B attached to the earth

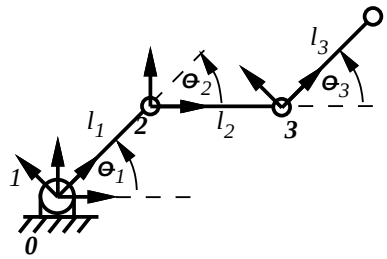
$$\begin{aligned} \boldsymbol{v}_A &= {}_A R_B(t)\boldsymbol{v}_B \\ \dot{\boldsymbol{v}}_A &= {}_A R_B[\dot{\boldsymbol{v}}_B + (\boldsymbol{\omega} \times \boldsymbol{v}_B)] \end{aligned}$$

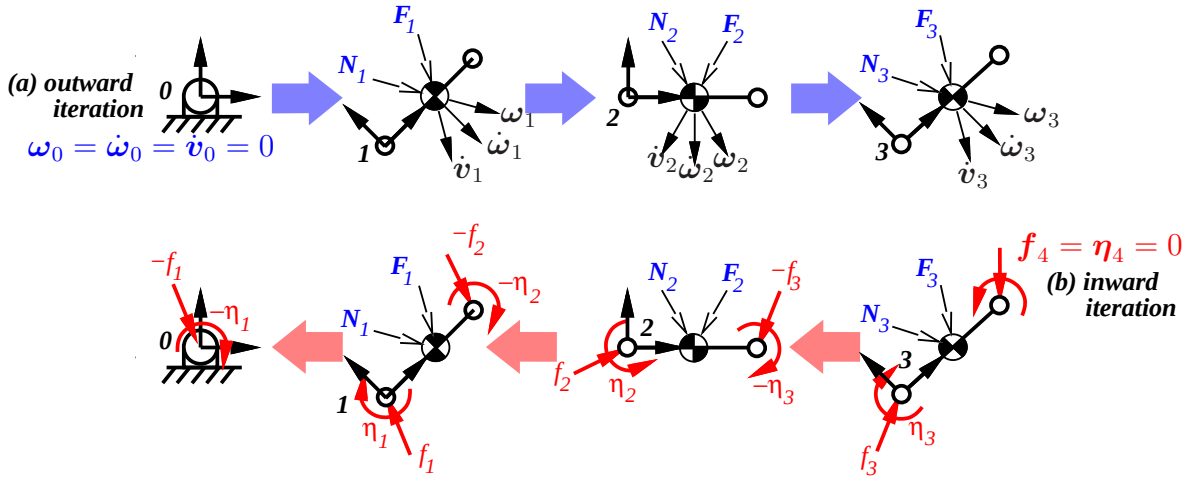
so that to an observer that travels with frame B:

$$\dot{\boldsymbol{v}}_B = {}_B R_A[\dot{\boldsymbol{v}}_A] - (\boldsymbol{\omega} \times \boldsymbol{v}_B)$$

a convergent flow and a rotating system, therefore, leads to a counterclockwise flow in the northern hemisphere and a clockwise rotation in the southern hemisphere.

Newton/Euler Method





the recursive equations for these iterations are derived in Appendix B of the book

Recursive Newton-Euler Equations

Outward Iterations

Angular Velocity: $\boldsymbol{\omega}$

REVOLUTE: $\quad {}^{i+1}\boldsymbol{\omega}_{i+1} = {}_{i+1}\mathbf{R}_i \; {}^{i}\boldsymbol{\omega}_i + \dot{\theta}_{i+1}\hat{\mathbf{z}}_{i+1}$

PRISMATIC: $\quad {}^{i+1}\boldsymbol{\omega}_{i+1} = {}_{i+1}\mathbf{R}_i \; {}^{i}\boldsymbol{\omega}_i$

Angular Acceleration: $\dot{\boldsymbol{\omega}}$

REVOLUTE: $\quad {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}_{i+1}\mathbf{R}_i \; {}^{i}\dot{\boldsymbol{\omega}}_i + ({}_{i+1}\mathbf{R}_i \; {}^{i}\boldsymbol{\omega}_i \times \dot{\theta}_{i+1}\hat{\mathbf{z}}_{i+1}) + \ddot{\theta}_{i+1}\hat{\mathbf{z}}_{i+1}$

PRISMATIC: $\quad {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}_{i+1}\mathbf{R}_i \; {}^{i}\dot{\boldsymbol{\omega}}_i$

Linear Acceleration: $\dot{\boldsymbol{v}}$

REVOLUTE: $\quad {}^{i+1}\dot{\boldsymbol{v}}_{i+1} = {}_{i+1}\mathbf{R}_i \left[{}^{i}\dot{\boldsymbol{v}}_i + ({}^{i}\dot{\boldsymbol{\omega}}_i \times {}^{i}\mathbf{p}_{i+1}) + ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{p}_{i+1}) \right]$

PRISMATIC: $\quad {}^{i+1}\dot{\boldsymbol{v}}_{i+1} = {}_{i+1}\mathbf{R}_i \left[{}^{i}\dot{\boldsymbol{v}}_i + \ddot{d}_i\hat{\mathbf{x}}_i + 2({}^{i}\boldsymbol{\omega}_i \times \dot{d}_i\hat{\mathbf{x}}_i) + ({}^{i}\dot{\boldsymbol{\omega}}_i \times d_i\hat{\mathbf{x}}_i) + ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{\omega}_i \times d_i\hat{\mathbf{x}}_i) \right]$

Linear Acceleration (center of mass): $\dot{\boldsymbol{v}}_{cm}$

$$
\begin{aligned}
{}^{i+1}\dot{\boldsymbol{v}}_{cm,(i+1)} = {} & ({}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} \times {}^{i+1}\mathbf{p}_{cm}) \\
& + ({}^{i+1}\boldsymbol{\omega}_{i+1} \times {}^{i+1}\boldsymbol{\omega}_{i+1} \times {}^{i+1}\mathbf{p}_{cm}) + {}^{i+1}\dot{\boldsymbol{v}}_{i+1}
\end{aligned}
$$

Net Force: F

$$ {}^{i+1}\mathbf{F}_{i+1} = m_{i+1} \; {}^{i+1}\dot{\boldsymbol{v}}_{cm \; i+1} $$

Net Moment: N

$$ {}^{i+1}\mathbf{N}_{i+1} = \boldsymbol{I}_{i+1} \; {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} + ({}^{i+1}\boldsymbol{\omega}_{i+1} \times \boldsymbol{I}_{i+1} \; {}^{i+1}\boldsymbol{\omega}_{i+1}) $$

Inward Iterations

Inter-Link Forces:

$$ {}^{i}\mathbf{f}_i = {}^{i}\mathbf{F}_i + {}_{i}\mathbf{R}_{i+1} \; {}^{i+1}\mathbf{f}_{i+1} $$

Inter-Link Moments:

$$
\begin{aligned}
{}^{i}\boldsymbol{\eta}_i = {} & {}^{i}\mathbf{N}_i + {}_{i}\mathbf{R}_{i+1} \; {}^{i+1}\boldsymbol{\eta}_{i+1} + ({}^{i}\mathbf{p}_{cm} \times {}^{i}\mathbf{F}_i) \\
& + ({}^{i}\mathbf{p}_{i+1} \times {}_{i}\mathbf{R}_{i+1} \; {}^{i+1}\mathbf{f}_{i+1})
\end{aligned}
$$

The Computed Torque Equation

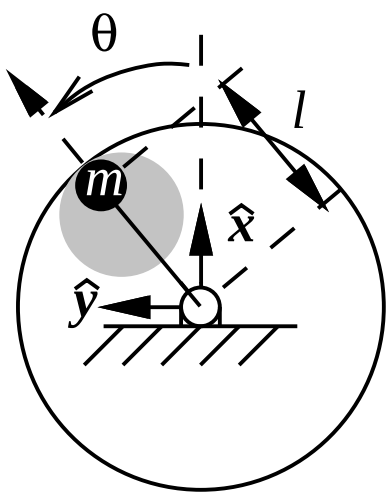
State Space Form

$$\boldsymbol{\tau} = \mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{V}(\boldsymbol{\theta}\ \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta}) + \mathbf{F}$$

external forces/torques:

- external forces
- friction
  - viscous $\tau = -v\dot{\theta}$
  - coulomb $\tau = -c(sgn(\dot{\theta}))$
  - hybrid

EXAMPLE: Dynamic Model of Roger's Eye



$$\sum \tau = \frac{d}{dt}(J\dot{\theta})$$

$$\tau_m + mglsin(\theta) = (ml^2)\ddot{\theta}$$

or

$$\tau_m = \mathbf{M}\ddot{\theta} + \mathbf{G},$$

generalized inertia

$$\mathbf{M} = ml^2 \text{ (a scalar);}$$

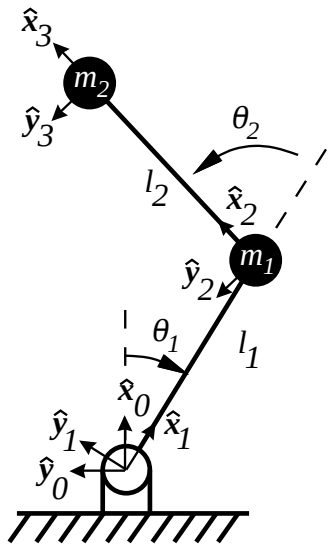
Coriolis and centripetal forces

$$\mathbf{V}(\theta, \dot{\theta}) \text{ do not exist; and}$$

Gravitational loads

$$\mathbf{G} = -mglsin(\theta)$$

EXAMPLE: Dynamic Model of Roger's Arm

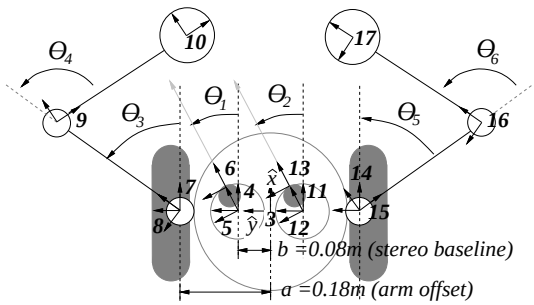


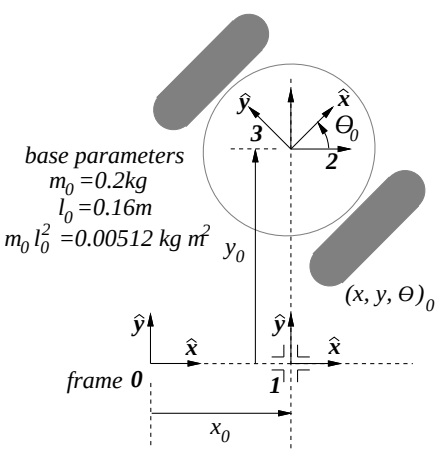
$$\mathbf{M}(\boldsymbol{\theta}) = \begin{bmatrix} m_2 l_2^2 + 2m_2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix}$$

$$\mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 (\dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2) \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix} \qquad Nm$$

$$\mathbf{G}(\boldsymbol{\theta}) = \begin{bmatrix} -(m_1 + m_2) l_1 s_1 g - m_2 l_2 s_{12} g \\ -m_2 l_2 s_{12} g \end{bmatrix} \qquad Nm$$

EXAMPLE: Roger's Whole-Body Dynamics





Roger's whole-body dynamics can also be written in the standardized form of the computed torque equation.

$$\widetilde{\boldsymbol{\tau}} = \mathbf{M}\ddot{\boldsymbol{q}} + \mathbf{V}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{F}$$

where $\widetilde{\boldsymbol{\tau}} \in \mathbb{R}^8$ is the vector of forces and torques causing acceleration in the degrees of freedom $\boldsymbol{q} \in \mathbb{R}^8$ of the robot.

$$\begin{bmatrix} \tilde{\tau}_0 \\ \tilde{\tau}_1 \\ \tilde{\tau}_2 \\ \tilde{\tau}_3 \\ \tilde{\tau}_4 \\ \tilde{\tau}_5 \\ \tilde{\tau}_6 \\ \tilde{\tau}_7 \end{bmatrix} = \begin{bmatrix} M_{00} & M_{01} & M_{02} & M_{03} & M_{04} & M_{05} & M_{06} & M_{07} \\ M_{10} & M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} & M_{17} \\ M_{20} & M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} & M_{27} \\ M_{30} & M_{31} & M_{32} & M_{33} & M_{34} & M_{35} & M_{36} & M_{37} \\ M_{40} & M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} & M_{47} \\ M_{50} & M_{51} & M_{52} & M_{53} & M_{54} & M_{55} & M_{56} & M_{57} \\ M_{60} & M_{61} & M_{62} & M_{63} & M_{64} & M_{65} & M_{66} & M_{67} \\ M_{70} & M_{71} & M_{72} & M_{73} & M_{74} & M_{75} & M_{76} & M_{77} \end{bmatrix} \begin{bmatrix} \ddot{q}_0 \\ \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \\ \ddot{q}_5 \\ \ddot{q}_6 \\ \ddot{q}_7 \end{bmatrix} + \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} + \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{3x} \\ \boldsymbol{\eta}_{3z} \\ \boldsymbol{\eta}_{5z} \\ \boldsymbol{\eta}_{12z} \\ \boldsymbol{\eta}_{8z} \\ \boldsymbol{\eta}_{9z} \\ \boldsymbol{\eta}_{15z} \\ \boldsymbol{\eta}_{16z} \end{bmatrix}$$

Simulation

$$\ddot{\boldsymbol{\theta}} = \mathbf{M}^{-1}(\boldsymbol{\theta})\left[\boldsymbol{\tau} - \mathbf{V}(\boldsymbol{\theta}\ \dot{\boldsymbol{\theta}}) - \mathbf{G}(\boldsymbol{\theta}) - \mathbf{F}\right]$$

initial conditions:

$$\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0 \qquad \dot{\boldsymbol{\theta}}(0) = \ddot{\boldsymbol{\theta}}(0) = 0$$

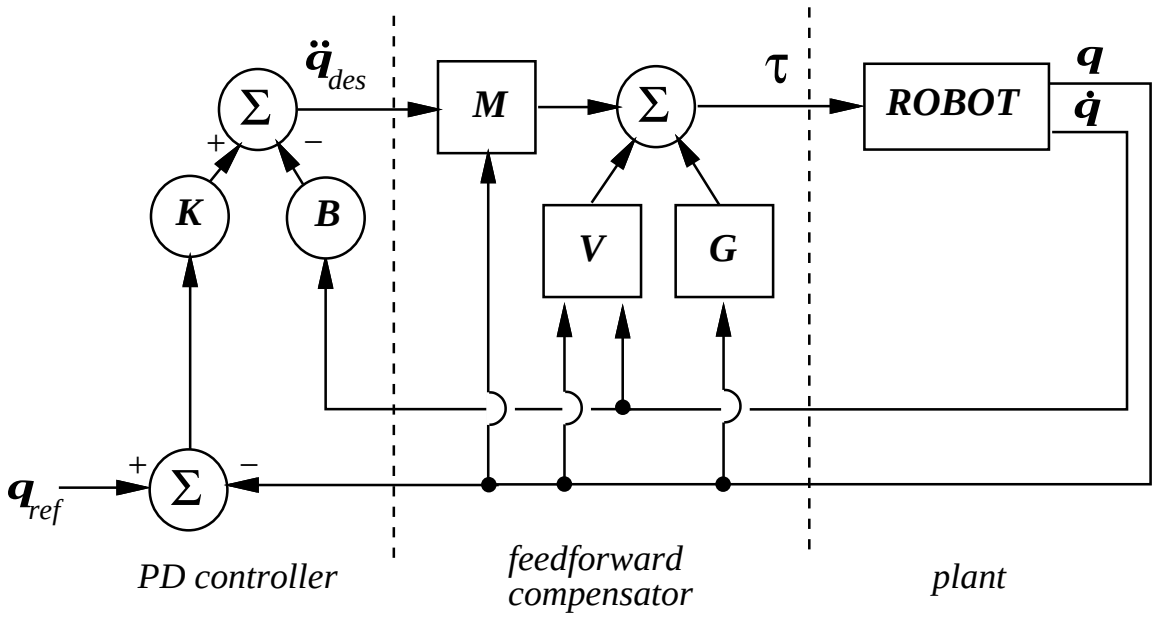
numerical integration:

$$\ddot{\boldsymbol{\theta}}(t) = \mathbf{M}^{-1}[\boldsymbol{\tau} - \mathbf{V} - \mathbf{G} - \mathbf{F}]$$

$$\dot{\boldsymbol{\theta}}(t + \Delta t) = \dot{\boldsymbol{\theta}}(t) + \ddot{\boldsymbol{\theta}}(t)\Delta t$$

$$\boldsymbol{\theta}(t + \Delta t) = \boldsymbol{\theta}(t) + \dot{\boldsymbol{\theta}}(t)\Delta t + \tfrac{1}{2}\ddot{\boldsymbol{\theta}}(t)\Delta t^2$$

Feedforward Dynamic Compensators



linearized and decoupled

Generalized Inertia Ellipsoid

computed torque equation:

$$\boldsymbol{\tau} = \mathbf{M}\ddot{\boldsymbol{\theta}} + \mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta})$$

if we assume that $\dot{\boldsymbol{\theta}} \approx 0$, and we ignore gravity

$$\boldsymbol{\tau} = \mathbf{M}\ddot{\boldsymbol{\theta}}$$

$$\|\ddot{\boldsymbol{\theta}}\| \leq 1$$

relative inertia—torque required to create a unit acceleration defined by the eigenvalues and eigenvectors of $\mathbf{M}\mathbf{M}^T$

Acceleration Polytope

gravity, actuator performance, and the current state of motion influences the ability of a manipulator to generate accelerations

differentiating $\dot{\boldsymbol{r}} = \mathbf{J}\dot{\boldsymbol{q}}$,

$$\begin{aligned}\ddot{\boldsymbol{r}} &= \mathbf{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\mathbf{J}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \\ &= \mathbf{J}\left[\mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{V} - \mathbf{G})\right] + \dot{\mathbf{J}}\dot{\boldsymbol{q}} \\ &= \mathbf{J}\mathbf{M}^{-1}\boldsymbol{\tau} + \dot{\boldsymbol{v}}_{vel} + \dot{\boldsymbol{v}}_{grav},\end{aligned}$$

$$\begin{aligned}\dot{\boldsymbol{v}}_{vel} &= -\mathbf{J}\mathbf{M}^{-1}\mathbf{V} + \dot{\mathbf{J}}\dot{\boldsymbol{q}}, \qquad \text{and} \\ \dot{\boldsymbol{v}}_{grav} &= -\mathbf{J}\mathbf{M}^{-1}\mathbf{G}.\end{aligned}$$

$$\tilde{\boldsymbol{\tau}} = \mathbf{L}^{-1}\boldsymbol{\tau} \qquad\qquad \mathbf{L} = diag({\tau_1}^{limit}, \ldots, {\tau_n}^{limit})$$

admissible torques constitute a unit hypercube $\|\tilde{\boldsymbol{\tau}}\|_\infty \leq 1$

$$\begin{aligned}\ddot{\boldsymbol{r}} &= \mathbf{J}\mathbf{M}^{-1}\mathbf{L}\tilde{\boldsymbol{\tau}} + \dot{\boldsymbol{v}}_{vel} + \dot{\boldsymbol{v}}_{grav} \\ &= \mathbf{J}\mathbf{M}^{-1}\mathbf{L}\tilde{\boldsymbol{\tau}} + \dot{\boldsymbol{v}}_{bias}.\end{aligned}$$

maps the n-dimensional hypercube $\|\tilde{\boldsymbol{\tau}}\|_\infty \leq 1$
to the m-dimensional *acceleration polytope*

Dynamic Manipulability Ellipsoid

$$\tilde{\boldsymbol{\tau}}^T\tilde{\boldsymbol{\tau}} = (\ddot{\boldsymbol{r}}-\dot{\boldsymbol{v}}_{bias})^T \left(\left[\mathbf{J}\mathbf{M}^{-1}\mathbf{L}\right]^{-1}\right)^T \left(\left[\mathbf{J}\mathbf{M}^{-1}\mathbf{L}\right]^{-1}\right)(\ddot{\boldsymbol{r}}-\dot{\boldsymbol{v}}_{bias}) \leq 1$$

$\mathbf{M}$ and $\mathbf{L}$ are symmetric:

$\mathbf{A}^{-T} = (\mathbf{A}^{-1})^T$, $\mathbf{A}^{-2} = \mathbf{A}^{-1}\mathbf{A}^{-1}$, and for symmetric matrices, $\mathbf{A}^T = \mathbf{A}$.

$$(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})^T \left[\mathbf{J}^{-T}\mathbf{M}\mathbf{L}^{-2}\mathbf{M}\mathbf{J}^{-1}\right](\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias}) \leq 1,$$

so that

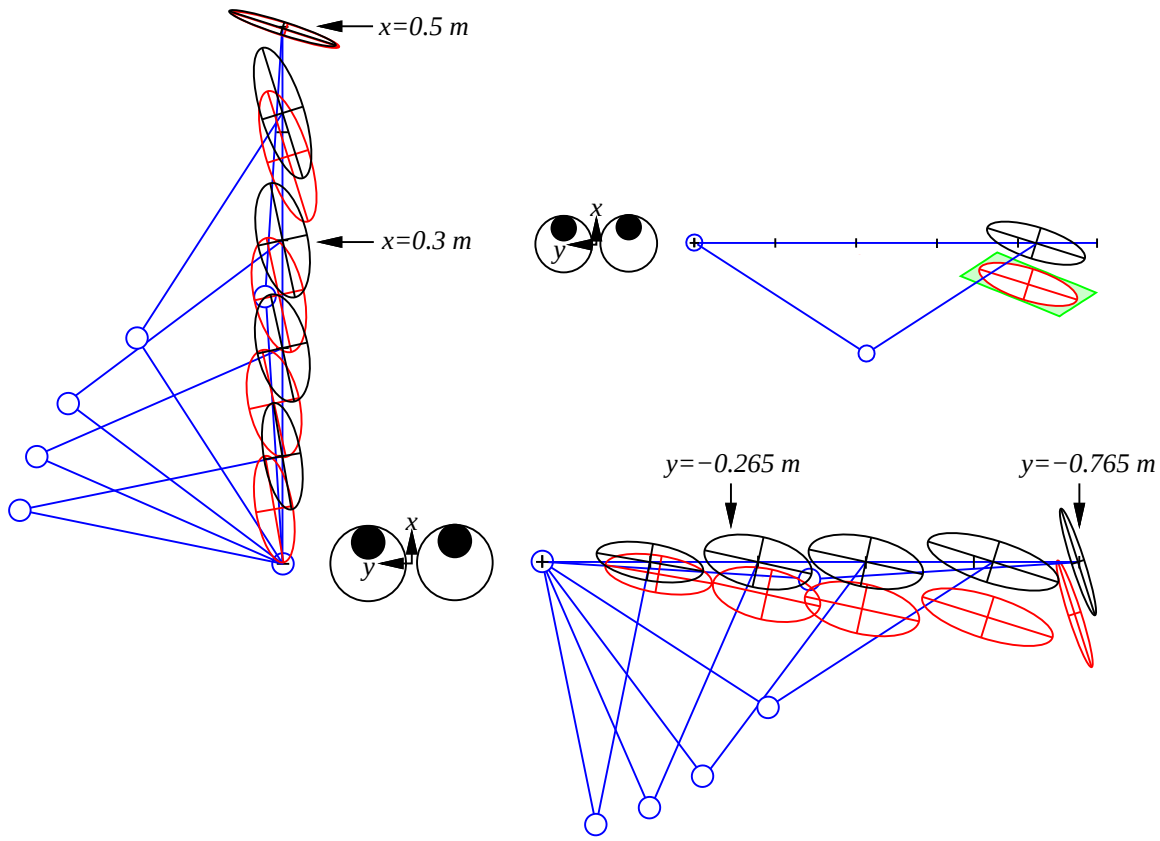
dynamic manipulability ellipsoid

$$(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})^T \in \left[\mathbf{J}\mathbf{M}^{-T}\mathbf{L}^2\mathbf{M}^{-1}\mathbf{J}^T\right]$$

dynamic-manipulability measure

$$\boldsymbol{\kappa}_d(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sqrt{det\left[\mathbf{J}(\mathbf{M}^T\mathbf{M})^{-1}\mathbf{J}^T\right]}$$

Conditioning Acceleration



$m_1 = m_2 = 0.2\ kg$, $l_1 = l_2 = 0.25\ m$, $\tau^T\tau \leq 0.005\ N^2m^2$.

black ellipsoids - unbiased dynamic manipulability
gravity biased dynamic manipulability
normalized acceleration polytope with gravity bias