Dynamics

The branch of physics that treats the action of force on bodies in motion or at rest; kinetics, kinematics, and statics, collectively. — Websters dictionary

Outline

- Conservation of Momentum
- Inertia Tensors - translation and rotation
- Newton/Euler Dynamics
- State Space, Configuration Space, and Cartesian Forms
- Lagrangian Dynamics
- Simulation and Feedforward Compensation

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Newton's Laws

1. the particle will remain in a state of constant rectilinear motion unless acted on by an external force;
2. the time-rate-of-change in the momentum (mv) of the particle is proportional to the externally applied forces, $F = \frac{d}{dt}(mv)$;
3. and any force imposed on body A by body B is reciprocated by an equal and opposite reaction force on body B by body A.

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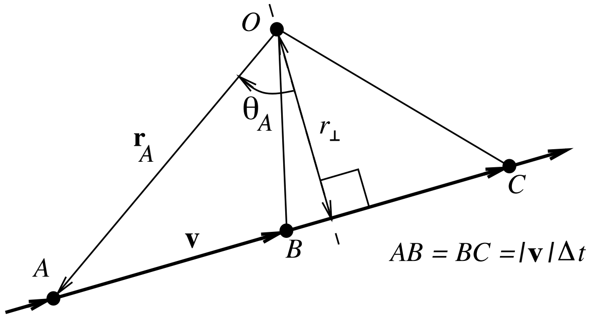
Conservation of Angular Momentum



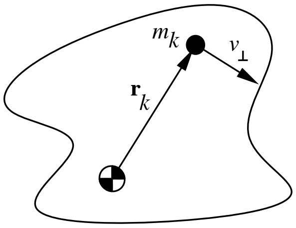
particle of mass m moving with uniform rectilinear velocity v ...

$$Area = \frac{1}{2} v r_\perp = \frac{1}{2} r cos(\theta) = constant$$

Similarly, the quantity, $\boxed{L = mrv_\perp}$ (angular momentum) is conserved for a particle that orbits a center of rotation, i.e. the particle sweeps out equal areas in equal times (Kepler's law of planetary motion).

For a collection of such particles:

$$L_{total} = \sum_k L_k = \sum_k m_k r_k v_{\perp,k} = \sum_k m_k r_k^2 \dot{\theta}_k$$



in the (rigid) planar lamina illustrated $\dot{\theta}_1 = \dot{\theta}_2 = \ldots = \dot{\theta}_k$

$$L_{total} = \left(\sum_k m_k r_k^2 \right) \dot{\theta}$$

$$I = \sum_k m_k r_k^2$$

(rotational moment of inertia)

$[kg - m^2]$

and

$$\tau = \frac{d}{dt} \left[I \dot{\theta} \right] = I \ddot{\theta}$$

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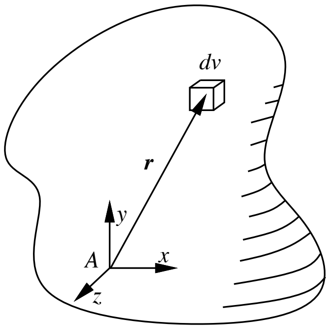
Inertia Tensor



MASS MOMENTS OF INERTIA	MASS PRODUCTS OF INERTIA
$I_{xx} = \int\int\int (y^2 + z^2)\rho dv$	$I_{xy} = \int\int\int xy\rho dv$
$I_{yy} = \int\int\int (x^2 + z^2)\rho dv$	$I_{xz} = \int\int\int xz\rho dv$
$I_{zz} = \int\int\int (x^2 + y^2)\rho dv$	$I_{yz} = \int\int\int yz\rho dv$

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

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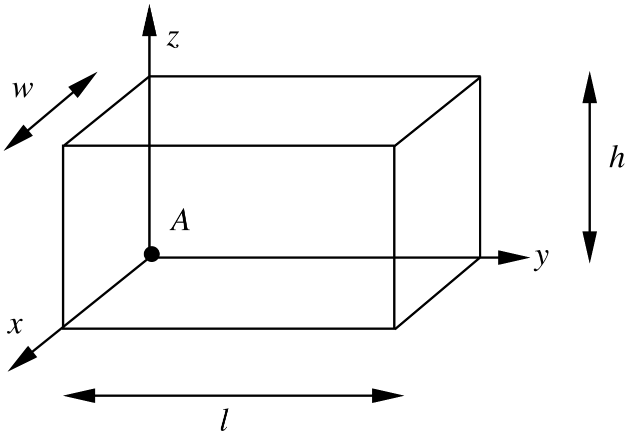
EXAMPLE: Inertia Tensor of Eccentric Rectangular Prism



$$
\begin{aligned}
I_{xx} &= \int_0^h \int_0^l \int_0^w (y^2+z^2)\rho dx dy dz \\
&= \int_0^h \int_0^l (y^2+z^2) w\rho dy dz \\
&= \int_0^h \left[(\frac{y^3}{3} + z^2 y) \right]_0^l w\rho dz \\
&= \int_0^h \left(\frac{l^3}{3} + z^2 l \right) w\rho dz \\
&= \left(\frac{l^3 z}{3} + \frac{l z^3}{3} \right) \Big|_0^h (w\rho) \\
&= \left(\frac{l^3 h}{3} + \frac{l h^3}{3} \right) w\rho
\end{aligned}
$$

or, since the mass of the rectangle $m = (wlh)\rho$,

$$I_{xx} = \frac{m}{3}(l^2 + h^2).$$

$$
{}^A I = \begin{bmatrix} \frac{m}{3}(l^2+h^2) & \frac{m}{4}wl & \frac{m}{4}hw \\ \frac{m}{4}wl & \frac{m}{3}(w^2+h^2) & \frac{m}{4}hl \\ \frac{m}{4}hw & \frac{m}{4}hl & \frac{m}{3}(l^2+w^2) \end{bmatrix}
$$

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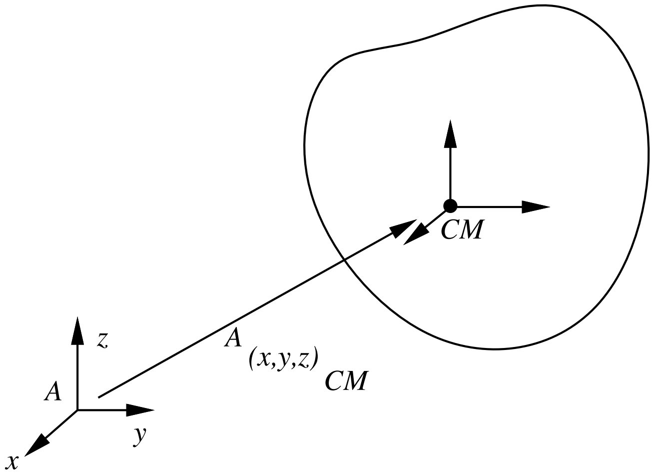
Parallel Axis Theorem - Translating the Inertia Tensor



the moments of inertia look like:

$$ {}^{A}I_{zz} = {}^{CM}I_{zz} + m({}^{A}x_{CM}^{2} + {}^{A}y_{CM}^{2}), $$

and the products of inertia are:

$$ {}^{A}I_{xy} = {}^{CM}I_{xy} + m({}^{A}x_{CM}\,{}^{A}y_{CM}). $$

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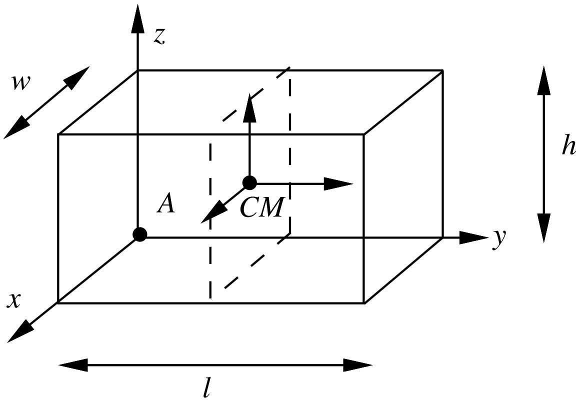
EXAMPLE: The Symmetric Rectangular Prism



$$
\begin{aligned}
{}^{CM}I_{zz} &= {}^{A}I_{zz} - m({}^{A}x_{CM}^2 + {}^{A}y_{CM}^2) \\
&= \frac{m}{3}(l^2 + w^2) - \frac{m}{4}(l^2 + w^2) \\
&= \frac{m}{12}(l^2 + w^2)
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{CM}I_{xy} &= {}^{A}I_{xy} - m({}^{A}x_{CM}\;{}^{A}y_{CM}) \\
&= \frac{m}{4}(wl) - \frac{m}{4}(wl) = 0.
\end{aligned}
$$

resulting in the diagonalized inertia tensor

$$
{}^{CM}I = \frac{m}{12}\begin{bmatrix} (l^2 + h^2) & 0 & 0 \\ 0 & (w^2 + h^2) & 0 \\ 0 & 0 & (l^2 + w^2) \end{bmatrix}
$$

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Rotating the Inertia Tensor

angular momentum $\mathbf{L}_0 = \mathbf{I}_0\boldsymbol{\omega}$ about frame 0 in a vector quantity that is conserved.

we can express it relative to frame 1 as

$$\mathbf{L}_1 = {}_1R_0\mathbf{L}_0$$

or

$$\begin{aligned} \mathbf{I}_1\boldsymbol{\omega}_1 &= R(\mathbf{I}_0\boldsymbol{\omega}_0) \\ &= R\mathbf{I}_0R^TR\boldsymbol{\omega}_0 \end{aligned}$$

and therefore,

$$\mathbf{I}_1 = R\mathbf{I}_0R^T.$$

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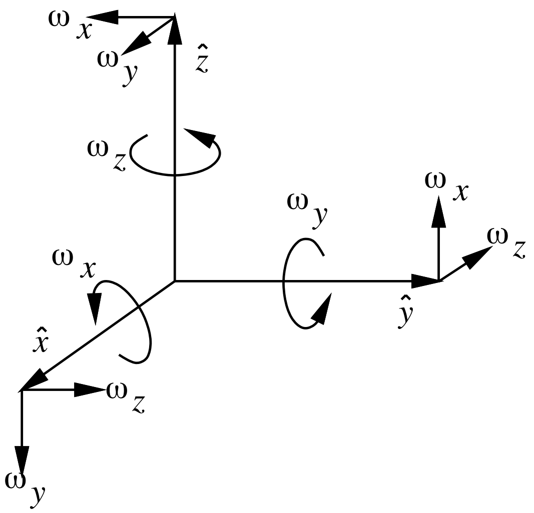
Rotating Coordinate Systems

Definition (Inertial Frame)

the frame where the absolute state of motion is completely known

Let frame A be an inertial frame. Frame B has an absolute velocity, $\boldsymbol{\omega}_B^B$ (written in terms of frame B coordinates).

$$\begin{aligned}
\mathbf{r}_A(t) &= \ {}_A R_B(t)\mathbf{r}_B(t) \\
\dot{\mathbf{r}}_A(t) &= \ {}_A R_B(t)\frac{d}{dt}[\mathbf{r}_B(t)] + \frac{d}{dt}[\ {}_A R_B(t)\ ]\mathbf{r}_B(t)
\end{aligned}$$



To evaluate the second term on the right, consider how the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, basis vectors for frame B change by virtue of $\boldsymbol{\omega}_B$.

$$\begin{array}{rcccc}
\dot{\hat{\mathbf{x}}} & = & & \omega_z\hat{\mathbf{y}} & -\omega_y\hat{\mathbf{z}} \\
\dot{\hat{\mathbf{y}}} & = & -\omega_z\hat{\mathbf{x}} & & -\omega_x\hat{\mathbf{z}} \\
\dot{\hat{\mathbf{z}}} & = & \omega_z\hat{\mathbf{y}} & -\omega_y\hat{\mathbf{z}} &
\end{array}$$

so

$$\begin{aligned}
\frac{d}{dt}[\ {}_A R_B(t)\ ]\mathbf{r}_B(t) &= \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \\
&= \boldsymbol{\omega} \times \mathbf{r}
\end{aligned}$$

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Rotating Coordinate Systems

Therefore,

$$\begin{aligned}\dot{\mathbf{r}}_A(t) &= {}_A R_B(t)\frac{d}{dt}[\mathbf{r}_B(t)] + \frac{d}{dt}[\ {}_A R_B(t)\ ]\mathbf{r}_B(t) \\ &= {}_A R_B\left[\dot{\mathbf{r}}_B + (\boldsymbol{\omega}_B^B \times \mathbf{r}_B)\right]\end{aligned}$$

and, in fact, all vector quantities expressed in local frames that are moving relative to an inertial frame are differentiated in this way

$$\frac{d}{dt}\left[{}_A\mathbf{R}_B(t)(\cdot)_B\right] =_A \mathbf{R}_B \left[\frac{d}{dt}(\cdot)_B + (\omega_B \times (\cdot)_B)\right]$$

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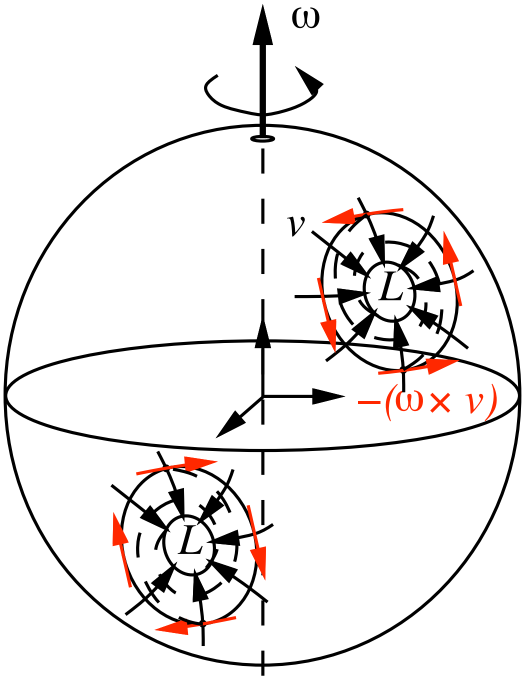
Rotating Coordinate Systems: Low Pressure Systems

Low pressure systems are regions in which large scale atmospheric flows converge. For the stationary (nonrotating) planet, this would result in flow lines directed radially inward.



But the earth rotates...
the atmospheric flow can be described in terms of a stationary inertial frame A and a rotating frame B attached to the earth

$$\begin{aligned}\mathbf{v}_A &= {}_AR_B(t)\mathbf{v}_B \\ \dot{\mathbf{v}}_A &= {}_AR_B[\dot{\mathbf{v}}_B + (\boldsymbol{\omega} \times \mathbf{v}_B)]\end{aligned}$$

so that an observer that travels with frame B:

$$\dot{\mathbf{v}}_B = {}_BR_A[\dot{\mathbf{v}}_A] - (\boldsymbol{\omega} \times \mathbf{v}_B)$$

...a convergent flow and a rotating system, therefore, leads to a counterclockwise flow in the northern hemisphere and a clockwise rotation in the southern hemisphere.

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