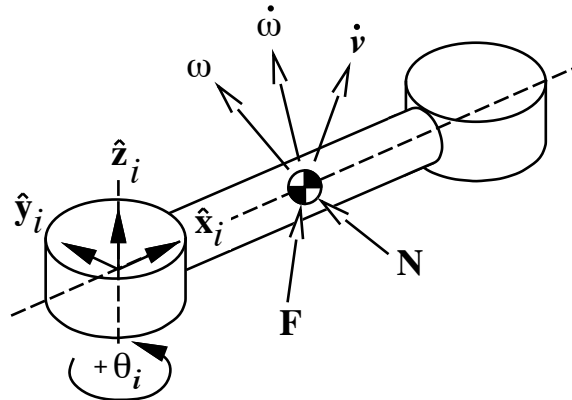


# Newton/Euler Equations



## Newton's Equation

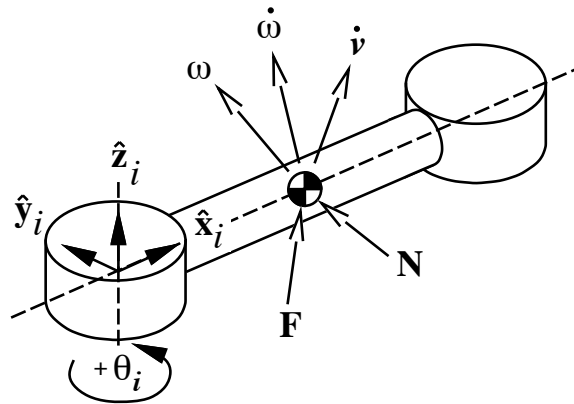
$$\begin{aligned}\mathbf{F} &= \frac{d}{dt} [R_i(m_i \mathbf{v}_i)] = R_i(m_i \dot{\mathbf{v}}_i) + \dot{R}_i(m_i \mathbf{v}_i) \\ &= R_i [m_i \dot{\mathbf{v}}_i + (\boldsymbol{\omega}_i \times m_i \mathbf{v}_i)]\end{aligned}$$

## Euler's Equation

$$\begin{aligned}\mathbf{N} &= \frac{d}{dt} [R_i(\mathbf{M}_i \boldsymbol{\omega}_i)] \\ &= R_i [\mathbf{M}_i \dot{\boldsymbol{\omega}}_i + (\boldsymbol{\omega}_i \times \mathbf{M}_i \boldsymbol{\omega}_i)]\end{aligned}$$

where  $\mathbf{F}$  and  $\mathbf{N}$  are the net force and torque vectors on link  $i$  written in inertial coordinates, and  $R_i$  is the rotation matrix relating frame  $i$  to the inertial frame, and  $\boldsymbol{\omega}_i$  is the total angular velocity of link  $i$  written in link  $i$  coordinates.

## Newton/Euler Equations



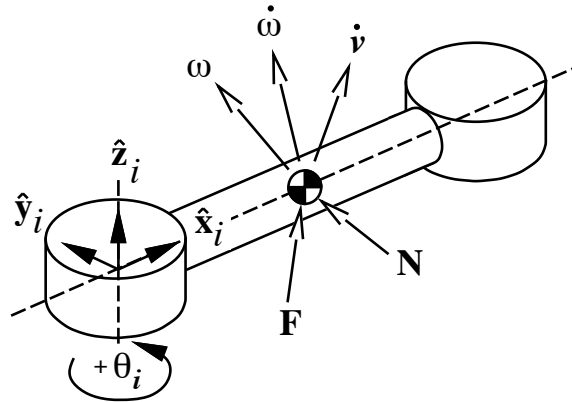
If  $F$  and  $N$  are written in the local coordinate frame for link  $i$ , then

$$\begin{bmatrix} m\mathbf{I}_3 & 0 \\ 0 & \mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\omega} \times m\mathbf{v} \\ \boldsymbol{\omega} \times \mathbf{M}\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{N} \end{bmatrix} = \mathbf{W}$$

$\mathbf{W} \in R^6$  is the generalized force or *wrench* consisting of forces and torques acting on link  $i$  written in link  $i$  coordinates.

...if we can account for the full state of motion,  $(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}, \dot{\mathbf{v}})$ , then we can compute the total load,  $\mathbf{W}$ , acting on the center of mass and define the equation of motion for link  $i$ .

## Recursive Newton/Euler Equations



Propagate the **absolute state of motion**,  $(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}, \dot{\mathbf{v}})_i$  at frame  $i$  to frame  $(i + 1)$ .

**Angular Velocity:  $\boldsymbol{\omega}$**

$$\text{REVOLUTE} : \quad {}^{i+1}\boldsymbol{\omega}_{i+1} = {}_{i+1}R_i \, {}^i\boldsymbol{\omega}_i + \dot{\theta}_{i+1} \hat{\mathbf{z}}_{i+1}$$

$$\text{PRISMATIC} : \quad {}^{i+1}\boldsymbol{\omega}_{i+1} = {}_{i+1}R_i \, {}^i\boldsymbol{\omega}_i$$

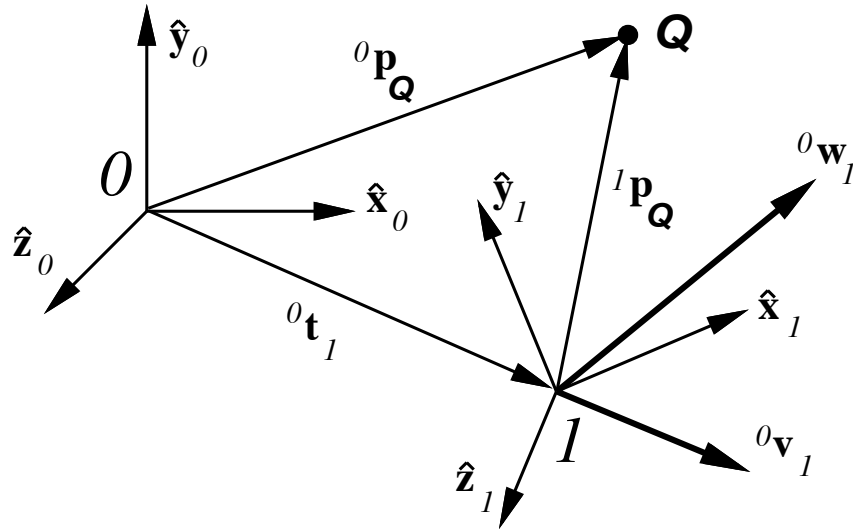
**Angular Acceleration:  $\dot{\boldsymbol{\omega}}$**

$$\text{REVOLUTE} : \quad {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}_{i+1}R_i \, {}^i\dot{\boldsymbol{\omega}}_i + ({}_{i+1}R_i \, {}^i\boldsymbol{\omega}_i \times \dot{\theta}_{i+1} \hat{\mathbf{z}}_{i+1}) \\ + \ddot{\theta}_{i+1} \hat{\mathbf{z}}_{i+1}$$

$$\text{PRISMATIC} : \quad {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}_{i+1}R_i \, {}^i\dot{\boldsymbol{\omega}}_i$$

## Recursive Newton/Euler Equations: cont.

Linear Acceleration:  $\dot{\mathbf{v}}$



$${}^0\mathbf{p}_Q = {}_0R_1 {}^1\mathbf{p}_Q + {}^0\mathbf{t}_1$$

$${}^0\mathbf{v}_Q = {}_0R_1 {}^1\dot{\mathbf{p}}_Q + ({}^0\boldsymbol{\omega}_1 \times {}_0R_1 {}^1\mathbf{p}_Q) + {}^0\mathbf{v}_1$$

$${}^0\dot{\mathbf{v}}_Q = \frac{d}{dt} [{}_0R_1 {}^1\dot{\mathbf{p}}_Q] + ({}^0\dot{\boldsymbol{\omega}}_1 \times {}_0R_1 {}^1\mathbf{p}_Q) + ({}^0\boldsymbol{\omega}_1 \times \frac{d}{dt} [{}_0R_1 {}^1\mathbf{p}_Q]) + {}^0\dot{\mathbf{v}}_1$$

$$= {}_0R_1 {}^1\ddot{\mathbf{p}}_Q + ({}^0\boldsymbol{\omega}_1 \times {}_0R_1 {}^1\dot{\mathbf{p}}_Q) + ({}^0\dot{\boldsymbol{\omega}}_1 \times {}_0R_1 {}^1\mathbf{p}_Q) + ({}^0\boldsymbol{\omega}_1 \times {}_0R_1 {}^1\dot{\mathbf{p}}_Q) + ({}^0\boldsymbol{\omega}_1 \times {}^0\boldsymbol{\omega}_1 \times {}_0R_1 {}^1\mathbf{p}_Q) + {}^0\dot{\mathbf{v}}_1$$

# Recursive Newton/Euler Equations: cont.

## Linear Acceleration: $\dot{\mathbf{v}}$

Now, substitute:

$$\text{frame } 0 \Leftrightarrow \text{frame } (i - 1)$$

$$\text{frame } 1 \Leftrightarrow \text{frame } i$$

$$\text{frame } 2 \Leftrightarrow \text{frame } (i + 1)$$

$$\begin{aligned} {}^{i+1}\dot{\mathbf{v}}_{i+1} &= {}_{i+1}R_{i-1} \left[ {}_{i-1}R_i {}^i\ddot{\mathbf{p}}_{i+1} + 2({}^{i-1}\boldsymbol{\omega}_i \times {}_{i-1}R_i {}^i\dot{\mathbf{p}}_{i+1}) \right. \\ &\quad \left. + ({}^{i-1}\dot{\boldsymbol{\omega}}_i \times {}_{i-1}R_i {}^i\mathbf{p}_{i+1}) \right. \\ &\quad \left. + ({}^{i-1}\boldsymbol{\omega}_i \times {}^{i-1}\boldsymbol{\omega}_i \times {}_{i-1}R_i {}^i\mathbf{p}_{i+1}) + {}^{i-1}\dot{\mathbf{v}}_i \right] \\ &= {}_{i+1}R_i \left[ {}^i\ddot{\mathbf{p}}_{i+1} + 2({}^i\boldsymbol{\omega}_i \times {}^i\dot{\mathbf{p}}_{i+1}) + ({}^i\dot{\boldsymbol{\omega}}_i \times {}^i\mathbf{p}_{i+1}) \right. \\ &\quad \left. + ({}^i\boldsymbol{\omega}_i \times {}^i\boldsymbol{\omega}_i \times {}^i\mathbf{p}_{i+1}) + {}^i\dot{\mathbf{v}}_i \right] \end{aligned}$$

$$\begin{aligned} \text{REVOLUTE : } & {}^i\mathbf{p}_{i+1} = \text{const}, \quad {}^i\dot{\mathbf{p}}_{i+1} = {}^i\ddot{\mathbf{p}}_{i+1} = 0 \\ & {}^{i+1}\dot{\mathbf{v}}_{i+1} = {}_{i+1}R_i \left[ {}^i\dot{\mathbf{v}}_i + ({}^i\dot{\boldsymbol{\omega}}_i \times {}^i\mathbf{p}_{i+1}) \right. \\ &\quad \left. + ({}^i\boldsymbol{\omega}_i \times {}^i\boldsymbol{\omega}_i \times {}^i\mathbf{p}_{i+1}) \right] \end{aligned}$$

$$\begin{aligned} \text{PRISMATIC : } & {}^i\mathbf{p}_{i+1} = d_i\hat{\mathbf{x}}_i, \quad {}^i\dot{\mathbf{p}}_{i+1} = \dot{d}_i\hat{\mathbf{x}}_i, \quad {}^i\ddot{\mathbf{p}}_{i+1} = \ddot{d}_i\hat{\mathbf{x}}_i \\ & {}^{i+1}\dot{\mathbf{v}}_{i+1} = {}_{i+1}R_i \left[ {}^i\dot{\mathbf{v}}_i + \ddot{d}_i\hat{\mathbf{x}}_i + 2({}^i\boldsymbol{\omega}_i \times \dot{d}_i\hat{\mathbf{x}}_i) \right. \\ &\quad \left. + ({}^i\dot{\boldsymbol{\omega}}_i \times d_i\hat{\mathbf{x}}_i) \right. \\ &\quad \left. + ({}^i\boldsymbol{\omega}_i \times {}^i\boldsymbol{\omega}_i \times d_i\hat{\mathbf{x}}_i) \right] \end{aligned}$$

## Recursive Newton/Euler Equations: cont.

Now, refer the translational acceleration to the center of mass:

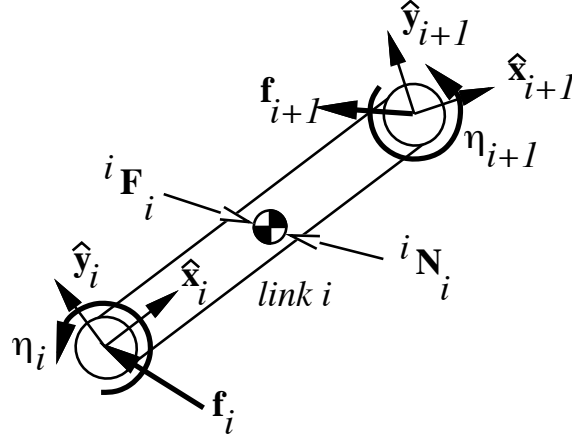
$${}^{i+1}\dot{\mathbf{v}}_{cm,(i+1)} = ({}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} \times {}^{i+1}\mathbf{p}_{cm}) + ({}^{i+1}\boldsymbol{\omega}_{i+1} \times {}^{i+1}\boldsymbol{\omega}_{i+1} \times {}^{i+1}\mathbf{p}_{cm}) + {}^{i+1}\dot{\mathbf{v}}_{i+1}$$

and we may write the Newton-Euler equation of motion:

$${}^{i+1}\mathbf{F}_{i+1} = m_{i+1} {}^{i+1}\dot{\mathbf{v}}_{cm\ i+1}$$

$${}^{i+1}\mathbf{N}_{i+1} = \mathbf{M}_{i+1} {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} + ({}^{i+1}\boldsymbol{\omega}_{i+1} \times \mathbf{M}_{i+1} {}^{i+1}\boldsymbol{\omega}_{i+1})$$

## Forces in Open Kinematic Chains



$$\sum Forces = {}^i\mathbf{F}_i = {}^i\mathbf{f}_i - {}_iR_{i+1} {}^{i+1}\mathbf{f}_{i+1}, \text{ or}$$

$${}^i\mathbf{f}_i = {}^i\mathbf{F}_i + {}_iR_{i+1} {}^{i+1}\mathbf{f}_{i+1}$$

$$\sum Torques = {}^i\mathbf{N}_i = {}^i\boldsymbol{\eta}_i - {}^i\boldsymbol{\eta}_{i+1} - ({}^i\mathbf{p}_{cm} \times {}^i\mathbf{f}_i) - (({}^i\mathbf{p}_{i+1} - {}^i\mathbf{p}_{cm}) \times {}^i\mathbf{f}_{i+1}),$$

but,  ${}^i\mathbf{f}_i = {}^i\mathbf{F}_i + {}_iR_{i+1} {}^{i+1}\mathbf{f}_{i+1}$ , so that,

$${}^i\mathbf{N}_i = {}^i\boldsymbol{\eta}_i - {}^i\boldsymbol{\eta}_{i+1} - ({}^i\mathbf{p}_{cm} \times {}^i\mathbf{F}_i) - ({}^i\mathbf{p}_{i+1} \times {}^i\mathbf{f}_{i+1})$$

or,

$${}^i\boldsymbol{\eta}_i = {}^i\mathbf{N}_i + {}_iR_{i+1} {}^{i+1}\boldsymbol{\eta}_{i+1} + ({}^i\mathbf{p}_{cm} \times {}^i\mathbf{F}_i) + ({}^i\mathbf{p}_{i+1} \times {}_iR_{i+1} {}^{i+1}\mathbf{f}_{i+1})$$