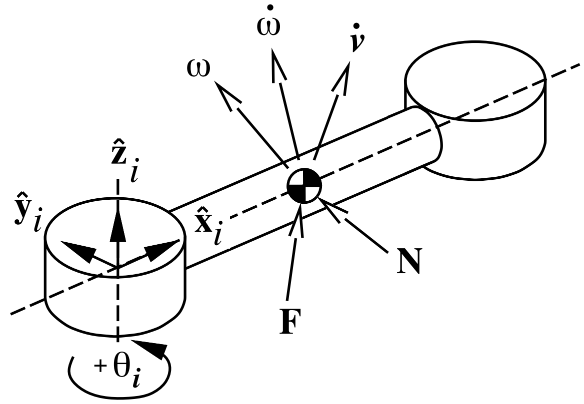
Newton/Euler Equations



Newton's Equation

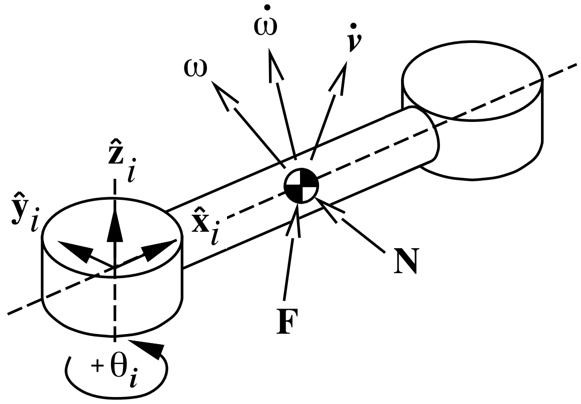
$$\begin{aligned}\mathbf{F} &= \frac{d}{dt}\left[R_i(m_i\mathbf{v}_i)\right] = R_i(m_i\dot{\mathbf{v}}_i) + \dot{R}_i(m_i\mathbf{v}_i) \\ &= R_i\left[m_i\dot{\mathbf{v}}_i + (\boldsymbol{\omega}_i \times m_i\mathbf{v}_i)\right]\end{aligned}$$

Euler's Equation

$$\begin{aligned}\mathbf{N} &= \frac{d}{dt}\left[R_i(\mathbf{M}_i\boldsymbol{\omega}_i)\right] \\ &= R_i\left[\mathbf{M}_i\dot{\boldsymbol{\omega}}_i + (\boldsymbol{\omega}_i \times \mathbf{M}_i\boldsymbol{\omega}_i)\right]\end{aligned}$$

where **F** and **N** are the net force and torque vectors on link i written in inertial coordinates, and R_i is the rotation matrix relating frame i to the inertial frame, and $\boldsymbol{\omega}_i$ is the total angular velocity of link i written in link i coordinates.

Newton/Euler Equations



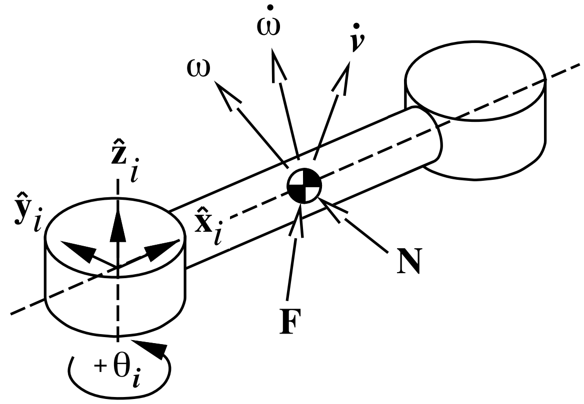
If F and N are written in the local coordinate frame for link i, then

$$\begin{bmatrix} m\mathbf{I}_3 & 0 \\ 0 & \mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\omega} \times m\mathbf{v} \\ \boldsymbol{\omega} \times \mathbf{M}\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{N} \end{bmatrix} = \mathbf{W}$$

$\mathbf{W} \in R^6$ is the generalized force or *wrench* consisting of forces and torques acting on link i written in link i coordinates.

...if we can account for the full state of motion, $(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}, \dot{\mathbf{v}})$, then we can compute the total load, $\mathbf{W}$, acting on the center of mass and define the equation of motion for link i.

Recursive Newton/Euler Equations



Propagate the absolute state of motion, $(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}, \dot{\mathbf{v}})_i$ at frame i to frame $(i+1)$.

Angular Velocity: $\boldsymbol{\omega}$

$$REVOLUTE: \quad {}^{i+1}\boldsymbol{\omega}_{i+1} = {}_{i+1}R_i \, {}^{i}\boldsymbol{\omega}_i + \dot{\theta}_{i+1}\hat{\mathbf{z}}_{i+1}$$

$$PRISMATIC: \quad {}^{i+1}\boldsymbol{\omega}_{i+1} = {}_{i+1}R_i \, {}^{i}\boldsymbol{\omega}_i$$

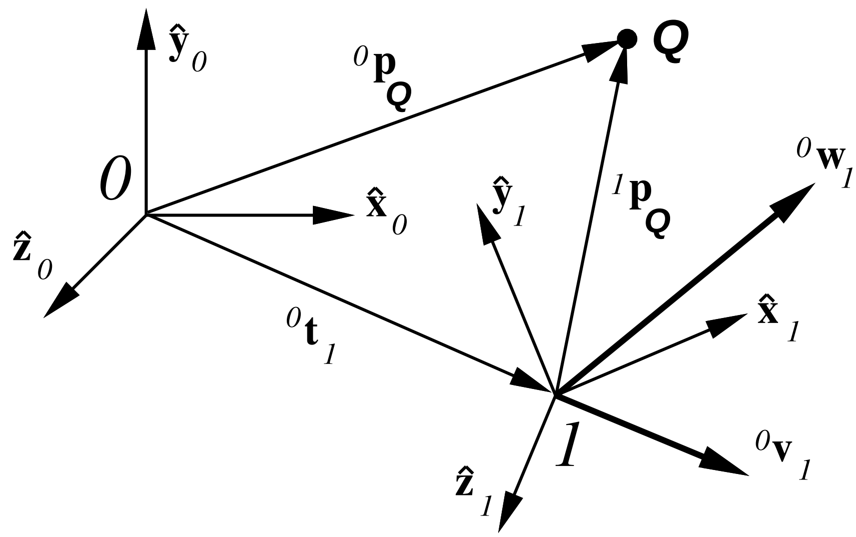
Angular Acceleration: $\dot{\boldsymbol{\omega}}$

$$REVOLUTE: \quad {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}_{i+1}R_i \, {}^{i}\dot{\boldsymbol{\omega}}_i + ({}_{i+1}R_i \, {}^{i}\boldsymbol{\omega}_i \times \dot{\theta}_{i+1}\hat{\mathbf{z}}_{i+1}) + \ddot{\theta}_{i+1}\hat{\mathbf{z}}_{i+1}$$

$$PRISMATIC: \quad {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}_{i+1}R_i \, {}^{i}\dot{\boldsymbol{\omega}}_i$$

Recursive Newton/Euler Equations: cont.

Linear Acceleration: $\dot{\mathbf{v}}$



$$^0\mathbf{p}_Q = {}_0R_1\ {}^1\mathbf{p}_Q + {}^0\mathbf{t}_1$$

$$^0\mathbf{v}_Q = {}_0R_1\ {}^1\dot{\mathbf{p}}_Q + ({}^0\boldsymbol{\omega}_1 \times {}_0R_1\ {}^1\mathbf{p}_Q) + {}^0\mathbf{v}_1$$

$$\begin{aligned} {}^0\dot{\mathbf{v}}_Q &= \frac{d}{dt}\left[{}_0R_1\ {}^1\dot{\mathbf{p}}_Q\right] + ({}^0\dot{\boldsymbol{\omega}}_1 \times {}_0R_1\ {}^1\mathbf{p}_Q) + ({}^0\boldsymbol{\omega}_1 \times \frac{d}{dt}\left[{}_0R_1\ {}^1\mathbf{p}_Q\right]) \\ &\quad + {}^0\dot{\mathbf{v}}_1 \\ &= {}_0R_1\ {}^1\ddot{\mathbf{p}}_Q + ({}^0\boldsymbol{\omega}_1 \times {}_0R_1\ {}^1\dot{\mathbf{p}}_Q) + ({}^0\dot{\boldsymbol{\omega}}_1 \times {}_0R_1\ {}^1\mathbf{p}_Q) \\ &\quad + ({}^0\boldsymbol{\omega}_1 \times {}_0R_1\ {}^1\dot{\mathbf{p}}_Q) + ({}^0\boldsymbol{\omega}_1 \times {}^0\boldsymbol{\omega}_1 \times {}_0R_1\ {}^1\mathbf{p}_Q) + {}^0\dot{\mathbf{v}}_1 \end{aligned}$$

Recursive Newton/Euler Equations: cont.

Linear Acceleration: $\dot{\mathbf{v}}$

Now, substitute:

$$\begin{aligned} frame\ 0 &\Leftrightarrow frame\ (i-1) \\ frame\ 1 &\Leftrightarrow frame\ i \\ frame\ 2 &\Leftrightarrow frame\ (i+1) \end{aligned}$$

$$\begin{aligned} {}^{i+1}\dot{\mathbf{v}}_{i+1} &= {}_{i+1}R_{i-1}\Big[\ {}_{i-1}R_i\ {}^{i}\ddot{\mathbf{p}}_{i+1} + 2(\ {}^{i-1}\boldsymbol{\omega}_i \times\ {}_{i-1}R_i\ {}^{i}\dot{\mathbf{p}}_{i+1}) \\ &\quad +(\ {}^{i-1}\dot{\boldsymbol{\omega}}_i \times\ {}_{i-1}R_i\ {}^{i}\mathbf{p}_{i+1}) \\ &\quad +(\ {}^{i-1}\boldsymbol{\omega}_i \times\ {}^{i-1}\boldsymbol{\omega}_i \times\ {}_{i-1}R_i\ {}^{i}\mathbf{p}_{i+1}) +\ {}^{i-1}\dot{\mathbf{v}}_i\Big] \\ &= {}_{i+1}R_i\Big[\ {}^{i}\ddot{\mathbf{p}}_{i+1} + 2(\ {}^{i}\boldsymbol{\omega}_i \times\ {}^{i}\dot{\mathbf{p}}_{i+1}) + (\ {}^{i}\dot{\boldsymbol{\omega}}_i \times\ {}^{i}\mathbf{p}_{i+1}) \\ &\quad +(\ {}^{i}\boldsymbol{\omega}_i \times\ {}^{i}\boldsymbol{\omega}_i \times\ {}^{i}\mathbf{p}_{i+1}) +\ {}^{i}\dot{\mathbf{v}}_i\Big] \end{aligned}$$

$$\begin{aligned} REVOLUTE: &\quad {}^{i}\mathbf{p}_{i+1} = const,\ {}^{i}\dot{\mathbf{p}}_{i+1} =\ {}^{i}\ddot{\mathbf{p}}_{i+1} = 0 \\ &\quad {}^{i+1}\dot{\mathbf{v}}_{i+1} = {}_{i+1}R_i\Big[\ {}^{i}\dot{\mathbf{v}}_i + (\ {}^{i}\dot{\boldsymbol{\omega}}_i \times\ {}^{i}\mathbf{p}_{i+1}) \\ &\qquad +(\ {}^{i}\boldsymbol{\omega}_i \times\ {}^{i}\boldsymbol{\omega}_i \times\ {}^{i}\mathbf{p}_{i+1})\Big] \end{aligned}$$

$$\begin{aligned} PRISMATIC: &\quad {}^{i}\mathbf{p}_{i+1} = d_i\hat{\mathbf{x}}_i,\ {}^{i}\dot{\mathbf{p}}_{i+1} = \dot{d}_i\hat{\mathbf{x}}_i,\ {}^{i}\ddot{\mathbf{p}}_{i+1} = \ddot{d}_i\hat{\mathbf{x}}_i \\ &\quad {}^{i+1}\dot{\mathbf{v}}_{i+1} = {}_{i+1}R_i\Big[\ {}^{i}\dot{\mathbf{v}}_i + \ddot{d}_i\hat{\mathbf{x}}_i + 2(\ {}^{i}\boldsymbol{\omega}_i \times \dot{d}_i\hat{\mathbf{x}}_i) \\ &\qquad +(\ {}^{i}\dot{\boldsymbol{\omega}}_i \times d_i\hat{\mathbf{x}}_i) \\ &\qquad +(\ {}^{i}\boldsymbol{\omega}_i \times\ {}^{i}\boldsymbol{\omega}_i \times d_i\hat{\mathbf{x}}_i)\Big] \end{aligned}$$

Recursive Newton/Euler Equations: cont.

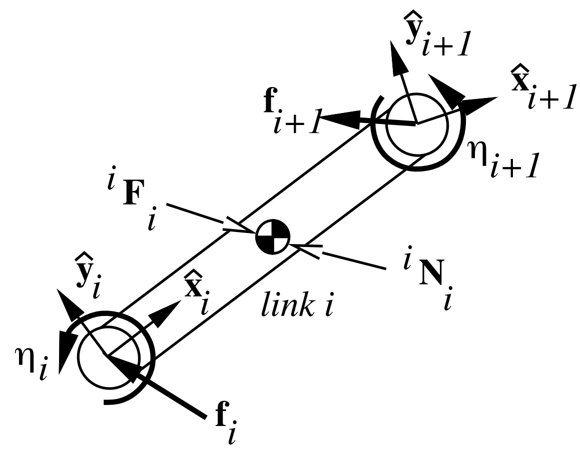
Now, refer the translational acceleration to the center of mass:

$$^{i+1}\dot{\mathbf{v}}_{cm,(i+1)} = (\,^{i+1}\dot{\boldsymbol{\omega}}_{i+1}\times\,^{i+1}\mathbf{p}_{cm})+(\,^{i+1}\boldsymbol{\omega}_{i+1}\times\,^{i+1}\boldsymbol{\omega}_{i+1}\times\,^{i+1}\mathbf{p}_{cm})+\,^{i+1}\dot{\mathbf{v}}_{i+1}$$

and we may write the Newton-Euler equation of motion:

$$\begin{aligned} ^{i+1}\mathbf{F}_{i+1} &= m_{i+1}\,^{i+1}\dot{\mathbf{v}}_{cm\ i+1} \\ ^{i+1}\mathbf{N}_{i+1} &= \mathbf{M}_{i+1}\,^{i+1}\dot{\boldsymbol{\omega}}_{i+1} + (\,^{i+1}\boldsymbol{\omega}_{i+1}\times\mathbf{M}_{i+1}\,^{i+1}\boldsymbol{\omega}_{i+1}) \end{aligned}$$

Forces in Open Kinematic Chains



$$\sum Forces \; = \; {}^{i}\mathbf{F}_i = {}^{i}\mathbf{f}_i - {}_{i}R_{i+1}\,{}^{i+1}\mathbf{f}_{i+1}, \; or$$

$$ {}^{i}\mathbf{f}_i = {}^{i}\mathbf{F}_i + {}_{i}R_{i+1}\,{}^{i+1}\mathbf{f}_{i+1}$$

$$\sum Torques = {}^{i}\mathbf{N}_i = {}^{i}\boldsymbol{\eta}_i - {}^{i}\boldsymbol{\eta}_{i+1} - ({}^{i}\mathbf{p}_{cm} \times {}^{i}\mathbf{f}_i) - (({}^{i}\mathbf{p}_{i+1} - {}^{i}\mathbf{p}_{cm}) \times {}^{i}\mathbf{f}_{i+1},$$

but, ${}^{i}\mathbf{f}_i = {}^{i}\mathbf{F}_i + {}_{i}R_{i+1}\,{}^{i+1}\mathbf{f}_{i+1}$, so that,

$${}^{i}\mathbf{N}_i = {}^{i}\boldsymbol{\eta}_i - {}^{i}\boldsymbol{\eta}_{i+1} - ({}^{i}\mathbf{p}_{cm} \times {}^{i}\mathbf{F}_i) - ({}^{i}\mathbf{p}_{i+1} \times {}^{i}\mathbf{f}_{i+1})$$

or,

$${}^{i}\boldsymbol{\eta}_i = {}^{i}\mathbf{N}_i + {}_{i}R_{i+1}\,{}^{i+1}\boldsymbol{\eta}_{i+1} + ({}^{i}\mathbf{p}_{cm} \times {}^{i}\mathbf{F}_i) + ({}^{i}\mathbf{p}_{i+1} \times {}_{i}R_{i+1}\,{}^{i+1}\mathbf{f}_{i+1})$$