Lagrangian Dynamics

Definition (Lagrangian) - The difference between the kinetic and potential energy of a dynamical system.

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

Theorem: (Lagrange's Equations) The equations of motion for a mechanical syste with generalized coordinates $q \in \mathcal{R}^m$ and Lagrangian, L are given by:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Upsilon_i \qquad i = 1, \ldots, m$$

where Υ_i is the vector of external forces acting on the i^{th} generalized coordinate, q_i.

In vector coordinates,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\mathbf{q}}$$

(proof: Calculus of Variations)

Lagrangian Dynamics - some intuition

if the Lagrangian is the difference between kinetic energy, $\frac{1}{2}m\dot{q}^2$ and the potential energy mgq

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} &= \frac{\partial L}{\partial q} + \Upsilon \\
\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{q}^2\right)\right] &= \frac{\partial}{\partial q}(mgq) + \Upsilon \\
\frac{d}{dt}(m\dot{q}) &= mg + \Upsilon \\
\frac{d}{dt}(momentum) &= applied\ force \quad \textbf{Newton's Equation}
\end{aligned}
$$

Simulation

$$\boldsymbol{\tau} = \mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{V}(\boldsymbol{\theta}\ \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta}) + \mathbf{F}$$

external forces/torques:

- external forces
- friction
  - viscous $\tau = -v\dot{\theta}$
  - coulomb $\tau = -c(sgn(\dot{\theta}))$
  - hybrid

$$\ddot{\boldsymbol{\theta}} = \mathbf{M}^{-1}(\boldsymbol{\theta})\left[\boldsymbol{\tau} - \mathbf{V}(\boldsymbol{\theta}\ \dot{\boldsymbol{\theta}}) - \mathbf{G}(\boldsymbol{\theta}) - \mathbf{F}\right]$$

initial conditions:

$$\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0 \qquad \dot{\boldsymbol{\theta}}(0) = \ddot{\boldsymbol{\theta}}(0) = 0$$

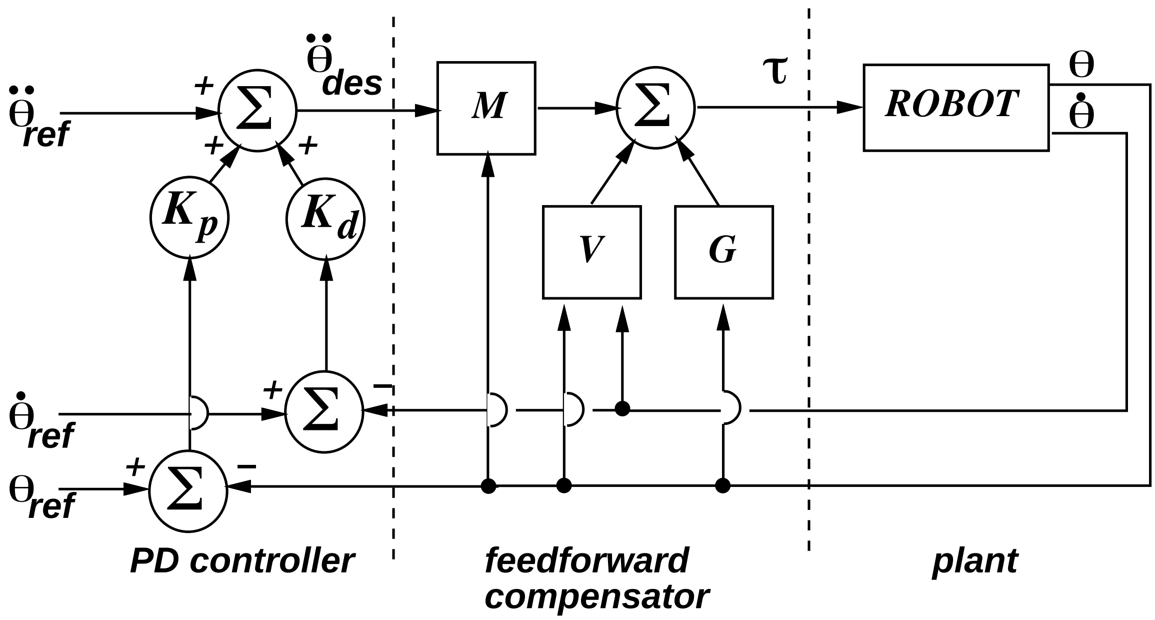
numerical integration:

$$\ddot{\boldsymbol{\theta}}(t) = \mathbf{M}^{-1}[\boldsymbol{\tau} - \mathbf{V} - \mathbf{G} - \mathbf{F}]$$

$$\dot{\boldsymbol{\theta}}(t + \Delta t) = \dot{\boldsymbol{\theta}}(t) + \ddot{\boldsymbol{\theta}}(t)\Delta t$$

$$\boldsymbol{\theta}(t + \Delta t) = \boldsymbol{\theta}(t) + \dot{\boldsymbol{\theta}}(t)\Delta t + \tfrac{1}{2}\ddot{\boldsymbol{\theta}}(t)\Delta t^2$$

Feedforward Dynamic Compensators



linearized and decoupled