Generalized Inertia Ellipsoid

computed torque equation:

$$\boldsymbol{\tau} = \mathbf{M}\ddot{\boldsymbol{\theta}} + \mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta})$$

if we assume that $\dot{\boldsymbol{\theta}} \approx 0$, and we ignore gravity

$$\boldsymbol{\tau} = \mathbf{M}\ddot{\boldsymbol{\theta}}$$

$$\|\ddot{\boldsymbol{\theta}}\| \leq 1$$

relative inertia—torque required to create a unit acceleration defined by the eigenvalues and eigenvectors of $\mathbf{M}\mathbf{M}^T$

Acceleration Polytope

gravity, actuator performance, and the current state of motion influences the ability of a manipulator to generate accelerations

differentiating $\dot{\boldsymbol{r}} = \mathbf{J}\dot{\boldsymbol{q}}$,

$$\begin{aligned}
\ddot{\boldsymbol{r}} &= \mathbf{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\mathbf{J}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \\
&= \mathbf{J}\left[\mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{V} - \mathbf{G})\right] + \dot{\mathbf{J}}\dot{\boldsymbol{q}} \\
&= \mathbf{J}\mathbf{M}^{-1}\boldsymbol{\tau} + \dot{\boldsymbol{v}}_{vel} + \dot{\boldsymbol{v}}_{grav},
\end{aligned}$$

$$\begin{aligned}
\dot{\boldsymbol{v}}_{vel} &= -\mathbf{J}\mathbf{M}^{-1}\mathbf{V} + \dot{\mathbf{J}}\dot{\boldsymbol{q}}, \qquad \text{and} \\
\dot{\boldsymbol{v}}_{grav} &= -\mathbf{J}\mathbf{M}^{-1}\mathbf{G}.
\end{aligned}$$

$$\tilde{\boldsymbol{\tau}} = \mathbf{L}^{-1}\boldsymbol{\tau} \qquad\qquad \mathbf{L} = diag({\tau_1}^{limit}, \ldots, {\tau_n}^{limit})$$

admissible torques constitute a unit hypercube $\|\tilde{\boldsymbol{\tau}}\|_\infty \leq 1$

$$\begin{aligned}
\ddot{\boldsymbol{r}} &= \mathbf{J}\mathbf{M}^{-1}\mathbf{L}\tilde{\boldsymbol{\tau}} + \dot{\boldsymbol{v}}_{vel} + \dot{\boldsymbol{v}}_{grav} \\
&= \mathbf{J}\mathbf{M}^{-1}\mathbf{L}\tilde{\boldsymbol{\tau}} + \dot{\boldsymbol{v}}_{bias}.
\end{aligned}$$

maps the n-dimensional hypercube $\|\tilde{\boldsymbol{\tau}}\|_\infty \leq 1$
to the m-dimensional *acceleration polytope*

Dynamic Manipulability Ellipsoid

$$\tilde{\boldsymbol{\tau}}^T\tilde{\boldsymbol{\tau}} = (\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})^T \left(\left[\mathbf{J}\mathbf{M}^{-1}\mathbf{L}\right]^{-1}\right)^T \left(\left[\mathbf{J}\mathbf{M}^{-1}\mathbf{L}\right]^{-1}\right)(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias}) \leq 1$$

$\mathbf{M}$ and $\mathbf{L}$ are symmetric:

$\mathbf{A}^{-T} = (\mathbf{A}^{-1})^T$, $\mathbf{A}^{-2} = \mathbf{A}^{-1}\mathbf{A}^{-1}$, and for symmetric matrices, $\mathbf{A}^T = \mathbf{A}$.

$$(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})^T \left[\mathbf{J}^{-T}\mathbf{M}\mathbf{L}^{-2}\mathbf{M}\mathbf{J}^{-1}\right](\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias}) \leq 1,$$

so that

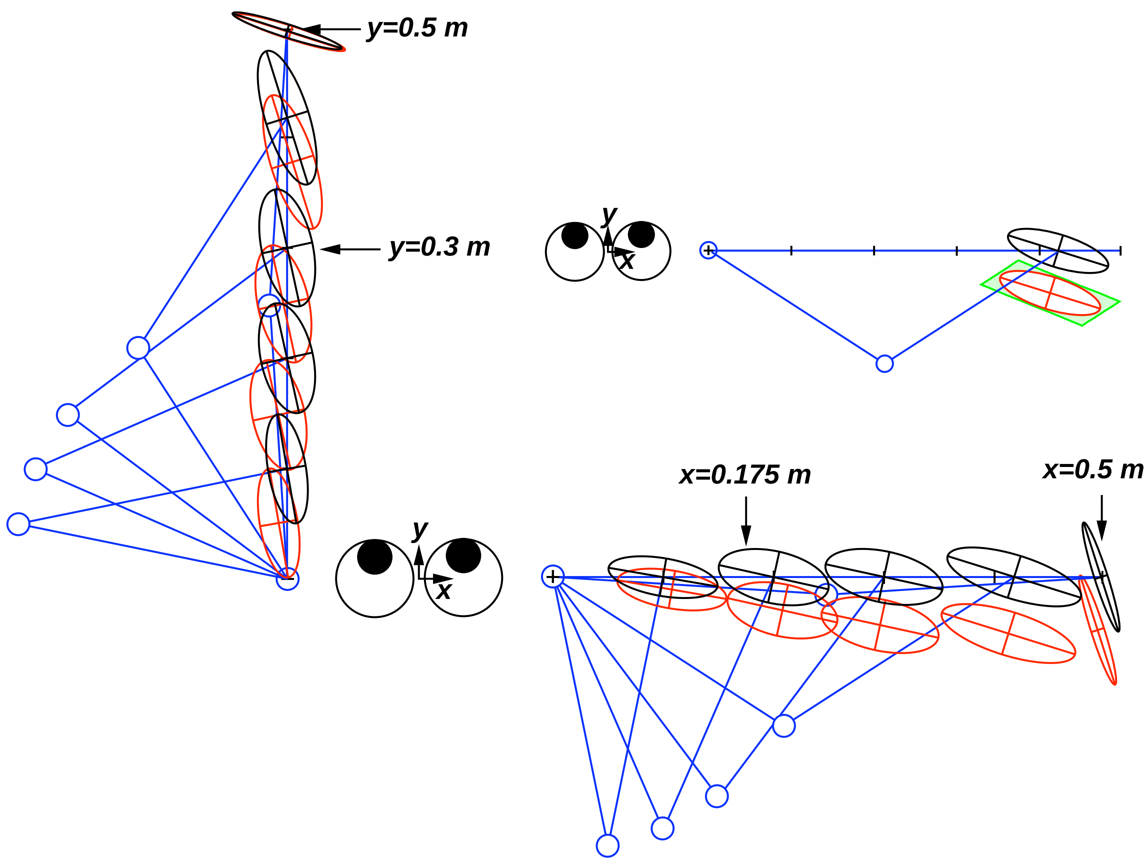
dynamic manipulability ellipsoid

$$(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})^T \in \left[\mathbf{J}\mathbf{M}^{-T}\mathbf{L}^2\mathbf{M}^{-1}\mathbf{J}^T\right]$$

dynamic-manipulability measure

$$\boldsymbol{\kappa}_d(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sqrt{det\left[\mathbf{J}(\mathbf{M}^T\mathbf{M})^{-1}\mathbf{J}^T\right]}$$

Conditioning Acceleration



$m_1 = m_2 = 0.2\ kg$, $l_1 = l_2 = 0.25\ m$, $\tau^T \tau \leq 0.005\ N^2 m^2$.

black ellipsoids - unbiased dynamic manipulability
gravity biased dynamic manipulability
normalized acceleration polytope with gravity bias

Velocity Affects

$$\begin{aligned}\ddot{\boldsymbol{x}} &= \mathbf{J}\mathbf{M}^{-1}\mathbf{L}\tilde{\boldsymbol{\tau}} + \dot{\boldsymbol{v}}_{vel} + \dot{\boldsymbol{v}}_{grav} \\ &= \mathbf{J}\mathbf{M}^{-1}\mathbf{L}\tilde{\boldsymbol{\tau}} + \dot{\boldsymbol{v}}_{bias}.\end{aligned}$$

maps the n-dimensional hypercube defined by $\|\tilde{\boldsymbol{\tau}}\|_\infty \leq 1$ to a m-dimensional polytope—feasible end-effector accelerations.