Newton-Euler Iteration: Summary
(Revolute Joints)

Outward Iteration

(A) $^{i+1}\bar{\omega}_{i+1} = {}_{i+1}R_i \, {}^{i}\bar{\omega}_i + \dot{\theta}\, {}^{i+1}\hat{z}_{i+1}$

(B) $^{i+1}\dot{\bar{\omega}}_{i+1} = {}_{i+1}R_i \, {}^{i}\dot{\bar{\omega}}_i + \left({}_{i+1}R_i \, {}^{i}\bar{\omega}_i \times \dot{\theta}_{i+1} \, {}^{i+1}\hat{z}_{i+1}\right) + \ddot{\theta}_{i+1} \, {}^{i+1}\hat{z}_{i+1}$

(C) $^{i+1}\dot{\bar{v}}_{i+1} = {}_{i+1}R_i \left[\left({}^{i}\dot{\bar{\omega}}_i \times {}^{i}\vec{P}_{i+1}\right) + \left({}^{i}\bar{\omega}_i \times \left({}^{i}\bar{\omega}_i \times {}^{i}\bar{P}_{i+1}\right)\right) + {}^{i}\dot{\bar{v}}_i \right]$

(D) $^{i+1}\dot{\bar{v}}_{cm_{i+1}} = \left({}^{i+1}\dot{\bar{\omega}}_{i+1} \times {}^{i+1}\bar{P}_{cm_{i+1}}\right) + \left({}^{i+1}\bar{\omega}_{i+1} \times \left({}^{i+1}\bar{\omega}_{i+1} \times {}^{i+1}\bar{P}_{cm_{i+1}}\right)\right) + {}^{i+1}\dot{\bar{v}}_{i+1}$

(E) $^{i+1}F_{i+1} = m_{i+1} \, {}^{i+1}\dot{\bar{v}}_{cm_{i+1}}$

(F) $^{i+1}N_{i+1} = {}^{i+1}I_{cm} \, {}^{i+1}\dot{\bar{\omega}}_{i+1} + \left({}^{i+1}\bar{\omega}_{i+1} \times {}^{i+1}I_{cm} \, {}^{i+1}\bar{\omega}_{i+1}\right)$

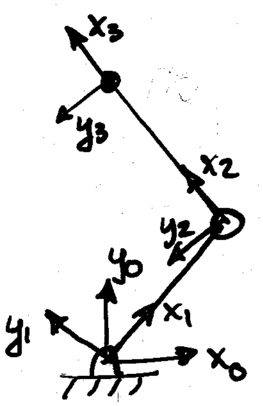
Inward Iteration

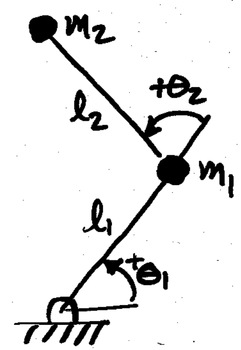
(A) $^{i}\bar{f}_i = {}^{i}F_i + {}_{i}R_{i+1} \, {}^{i+1}\bar{f}_{i+1}$

(B) $^{i}\bar{\eta}_i = {}^{i}N_i + {}_{i}R_{i+1} \, {}^{i+1}\bar{\eta}_{i+1} + \left({}^{i}\bar{P}_{cm} \times {}^{i}F_i\right) + \left({}^{i}\bar{P}_{i+1} \times {}_{i}R_{i+1} \, {}^{i+1}f_{i+1}\right)$

EXAMPLE:

2 DOF PLANAR MANIPULATOR





LUMPED POINT MASS AT DISTAL END OF EACH LINK

$\Downarrow$

$${}^1I_{cm} = {}^2I_{cm} = 0$$

$${}^1I_1 = m_1 l_1^2$$

$${}^2I_2 = m_2 l_2^2$$

$${}^1\bar{P}_2 = l_1 \hat{x}_1 \qquad {}^1\bar{P}_{cm} = l_1 \hat{x}_1$$

$${}^2\bar{P}_3 = l_2 \hat{x}_2 \qquad {}^2P_{cm} = l_2 \hat{x}_2$$

$${}_iR_{i+1} = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0 \\ s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}_{i+1}R_i = \begin{bmatrix} c_{i+1} & s_{i+1} & 0 \\ -s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

INITIAL CONDITIONS:

OUTWARD: $\quad {}^0\bar{\omega}_0 = {}^0\dot{\bar{\omega}}_0 = 0$

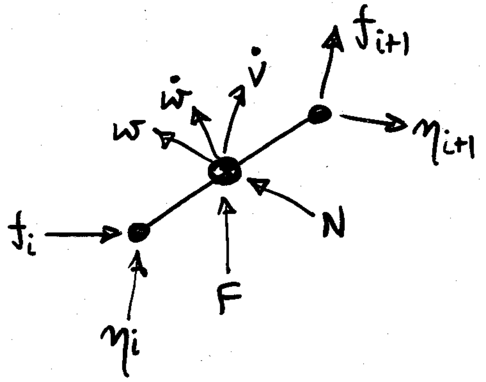
$${}^0\bar{V}_0 = 0 \qquad {}^0\dot{\bar{V}}_0 = G\,{}^0\hat{g}_0 \qquad \text{*GRAVITY*}$$

INWARD: $\quad {}^3\bar{f}_3 = {}^3\bar{\eta}_3 = 0$

(${}^3\bar{f}_3$: *force exerted* <u>on</u> *link 3* <u>by</u> *link 2*)

EXAMPLE: 2 DOF PLANAR MANIPULATOR

LINK #1



(A) $${}^1\bar{\omega}_1 = {}_1R_0\,{}^0\cancel{\bar{\omega}_0} + \dot{\theta}_1\,{}^1\hat{z}_1 = (0\ \ 0\ \ \dot{\theta}_1)^T$$

(B) $${}^1\dot{\bar{\omega}}_1 = {}_1R_0\,{}^0\cancel{\dot{\bar{\omega}}_0} + \left({}_1R_0\,{}^0\cancel{\bar{\omega}_0} \times \dot{\theta}_1\,{}^1\hat{z}_1\right) + \ddot{\theta}_1\,{}^1\hat{z}_1 = (0\ \ 0\ \ \ddot{\theta}_1)^T$$

(C) $${}^1\dot{\bar{V}}_1 = {}_1R_0\left[\left({}^0\cancel{\dot{\bar{\omega}}_0} \times {}^0\bar{P}_1\right) + \left({}^0\cancel{\bar{\omega}_0} \times {}^0\cancel{\bar{\omega}_0} \times {}^0\bar{P}_1\right) + {}^0\dot{\bar{V}}_0\right] = {}_1R_0\,{}^0\dot{\bar{V}}_0$$

$$\begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = (gs_1\ \ gc_1\ \ 0)^T$$

(D) $${}^1\dot{\bar{V}}_{cm} = \left({}^1\dot{\bar{\omega}}_1 \times {}^1\bar{P}_{cm}\right) + \left({}^1\bar{\omega}_1 \times {}^1\bar{\omega}_1 \times {}^1\bar{P}_{cm}\right) + {}^1\dot{\bar{V}}_1$$

$$= \begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ l_1 & 0 & 0 \end{vmatrix} + {}^1\bar{\omega}_1 \times \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ l_1 & 0 & 0 \end{vmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & l_1\dot{\theta}_1 & 0 \end{vmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 - l_1\dot{\theta}_1^2 \\ gc_1 + l_1\ddot{\theta}_1 \\ 0 \end{bmatrix}$$

(E) $${}^1F_1 = m_1\,{}^1\dot{\bar{V}}_{cm} = m_1\begin{bmatrix} gs_1 - l_1\dot{\theta}_1^2 \\ gc_1 + l_1\ddot{\theta}_1 \\ 0 \end{bmatrix}$$

(F) $${}^1N_1 = {}^1\cancel{I_{cm}}\,{}^1\dot{\bar{\omega}}_1 + \left({}^1\bar{\omega}_1 \times \overset{0}{\cancel{{}^1I_{cm}}}\,{}^1\bar{\omega}_1\right) = (0\ \ 0\ \ 0)^T$$

EXAMPLE: 2 DOF PLANAR MANIPULATOR

LINK #2

(A) $${}^2\bar{\omega}_2 = {}_2R_1\,{}^1\bar{\omega}_1 + \dot{\theta}_2\,\hat{z}_2 = \begin{bmatrix} c2 & s2 & 0 \\ -s2 & c2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

(B) $${}^2\dot{\bar{\omega}}_2 = {}_2R_1\,{}^1\dot{\bar{\omega}}_1 + ({}_2R_1\,{}^1\bar{\omega}_1 \times \dot{\theta}_2\,\hat{z}_2) + \ddot{\theta}_2\,\hat{z}_2$$

$$= \begin{bmatrix} c2 & s2 & 0 \\ -s2 & c2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} c2 & s2 & 0 \\ -s2 & c2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & 0 & \dot{\theta}_2 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

(C) $${}^2\dot{v}_2 = {}_2R_1\left[({}^1\dot{\bar{\omega}}_1 \times {}^1\bar{P}_2) + ({}^1\bar{\omega}_1 \times {}^1\bar{\omega}_1 \times {}^1P_2) + {}^1\dot{v}_1\right]$$

$$= \begin{bmatrix} c2 & s2 & 0 \\ -s2 & c2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left[\begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ l_1 & 0 & 0 \end{vmatrix} + {}^1\bar{\omega}_1 \times \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ l_1 & 0 & 0 \end{vmatrix} + \begin{bmatrix} g s1 \\ g c1 \\ 0 \end{bmatrix}\right]$$

$$= \begin{bmatrix} c2 & s2 & 0 \\ -s2 & c2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -l_1\dot{\theta}_1^2 + g s1 \\ l_1\ddot{\theta}_1 + g c1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\ddot{\theta}_1 s2 - l_1\dot{\theta}_1^2 c2 + g s12 \\ l_1\ddot{\theta}_1 c2 + l_1\dot{\theta}_1^2 s2 + g c12 \\ 0 \end{bmatrix}$$

EXAMPLE: 2 DOF PLANAR MANIPULATOR

OUTWARD
LINK #2 (CONT.)

$$(D)\quad {}^2\dot{\bar{v}}_{cm_2} = ({}^2\dot{\bar{\omega}}_2 \times {}^2\bar{P}_{cm}) + ({}^2\bar{\omega}_2 \times {}^2\bar{\omega}_2 \times {}^2\bar{P}_{cm}) + {}^2\dot{\bar{v}}_2$$

$$= \begin{bmatrix} -l_2(\dot{\theta}_1+\dot{\theta}_2)^2 + l_1\ddot{\theta}_1 s2 - l_1\dot{\theta}_1^2 c2 + g s12 \\ l_2(\ddot{\theta}_1+\ddot{\theta}_2) + l_1\ddot{\theta}_1 c2 + l_1\dot{\theta}_1^2 s2 + g c12 \\ 0 \end{bmatrix}$$

$$(E)\quad {}^2F_2 = m_2\, {}^2\dot{\bar{v}}_{cm_2}$$

$$(F)\quad {}^2N_2 = \bar{0}$$

INWARD
LINK #2

$$(A)\quad {}^2f_2 = {}_2R_3\, \overset{0}{\cancel{{}^3f_3}} + {}^2F_2$$

$$= {}^2F_2 = m_2\, {}^2\dot{\bar{v}}_{cm_2}$$

$$(B)\quad {}^2n_2 = \cancel{{}^2N_2} + {}_2R_3\, \cancel{{}^3n_3} + ({}^2P_{cm_2} \times {}^2F_2) + ({}^2P_3 \times {}_2R_3\, \cancel{{}^3f_3})$$

$$= \begin{bmatrix} 0 \\ 0 \\ l_2^2 m_2(\ddot{\theta}_1+\ddot{\theta}_2) + l_1 l_2 m_2 \ddot{\theta}_1 c_2 + l_1 l_2 m_2 \dot{\theta}_1^2 s_2 + l_2 m_2 g c12 \end{bmatrix}$$

τ_2: torque causing acceleration in link 2 about $\hat{z}_2$ axis

EXAMPLE: 2 DOF PLANAR MANIPULATOR

INWARD LINK #1

(A) $${}^1\bar{f}_1 = {}_1R_2\,{}^2\bar{f}_2 + {}^1F_1$$

WE HAVE ALL OF THESE TERMS, BUT WE DON'T NEED ${}^1f_1$ FOR ANYTHING, SO WE'LL SKIP IT

(B) $${}^1n_1 = {}^1N_1 + {}_1R_2\,{}^2\bar{n}_2 + ({}^1\bar{P}_{cm_1} \times {}^1F_1) + ({}^1P_2 \times {}_1R_2\,{}^2f_2)$$

$$\vdots$$

$$= \begin{bmatrix} 0 \\ 0 \\ m_2 l_2^2(\ddot{\theta}_1+\ddot{\theta}_2) + m_2 l_2 l_1 c_2(2\ddot{\theta}_1+\ddot{\theta}_2) + (m_1+m_2) l_1^2 \ddot{\theta}_1 \\ - m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + l_2 m_2 g c_{12} \\ + (m_1+m_2) l_1 g c_1 \end{bmatrix}$$

τ_1 : torque causing acceleration in link 1 about $\hat{z}_1$ axis

STRUCTURE OF THE DYNAMIC EQUATIONS

STATE SPACE FORM $\qquad \tau = \underset{2\times2}{M(\theta)}\,\ddot{\theta} + \underset{2\times1}{V(\theta\dot{\theta})} + \underset{2\times1}{G(\theta)}$

$M(\theta)$: 2x2, configuration dependent inertia matrix

$V(\theta\dot{\theta})$: 2x1 vector of centrifugal and coriolis terms

$G(\theta)$: 2x1 vector of gravity terms

FOR OUR 2DOF PLANAR ARM:

$$M(\theta) = \begin{bmatrix} m_2 l_2^2 + 2 m_2 l_1 l_2 c_2 + (m_1+m_2) l_1^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix}$$

$$V(\theta\dot{\theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ l_1 l_2 m_2 \dot{\theta}_1^2 s_2 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} l_2 m_2 g c_{12} + (m_1+m_2) l_1 g c_1 \\ l_2 m_2 g c_{12} \end{bmatrix}$$