Kinematics

> *A branch of dynamics that deals with aspects of motion apart from considerations of force and mass* — Websters dictionary

links The individual rigid bodies that collectively form a robot device.

joints Links are connected in pairs through revolute of prismatic constraints. A prismatic joint is a degree of freedom with a controllable length specifying the relative position of a slider and a guide link. Typically the guide link is linear so that the slider executes a straight line motion. A revolute joint connects two links through a rotational bearing. This degree of freedom is typically about a fixed axis of rotation and the two links, therefore, it executes motions within the plane defined by the links.

kinematic chain An assemblage of interconnected links.

mechanism When one of the links is held fixed (ground link) and the others move relative to the fixed link.

closed chain Kinematic chains with every link connected through joints to two adjacent links.

Kinematics (cont.)

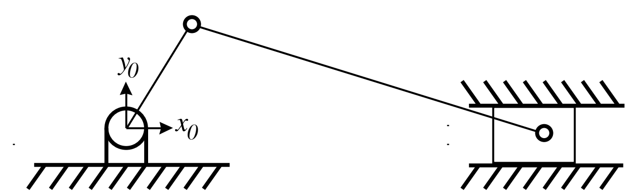
open chain A link may only be connected to one joint (unitary link).

configuration variable Any parameter (length or angle) of an underconstrained mechanism that is necessary to uniquely determine the configuration of the mechanism.

configuration space Devices for which multiple configuration variables must be specified are described in a configuration space.

degrees of freedom The minimum number of configuration variables necessary to fully define the configuration of a mechanism.

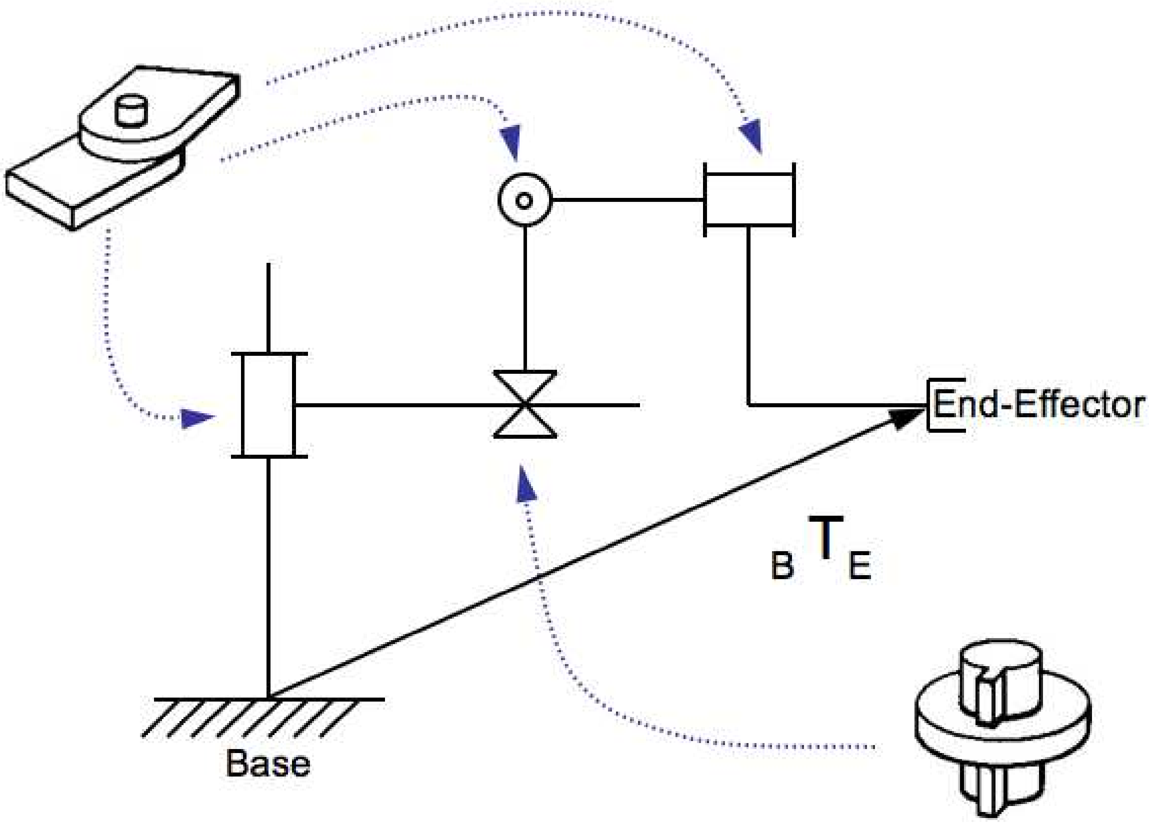
Example - A Familiar Mechanism



For the piston-cylinder-crank mechanism illustrated:

1. how many links does it have?
2. how many degrees of freedom does it have?
3. how many joints does it have?

Schematic Diagrams of Open-Chain Mechanisms



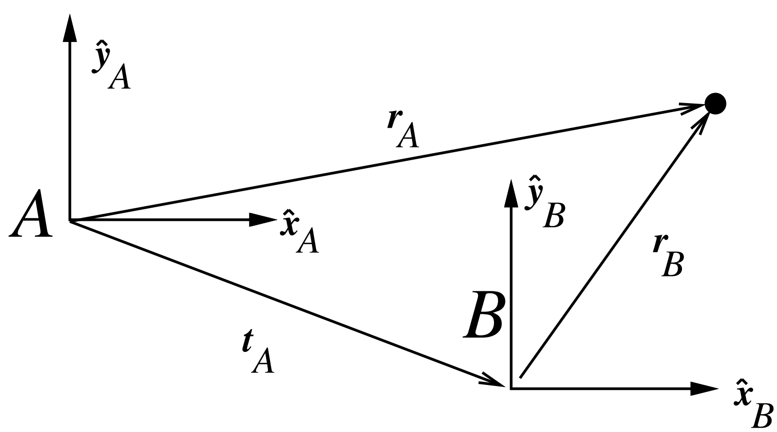
Spatial Relationships

Free Bodies

A *free body* has 6 spatial degrees of freedom:

$$\text{translations: } \vec{t} \in \mathcal{R}^{\ni}$$
$$\text{rotations: } \mathbf{R} \in SO(3)$$

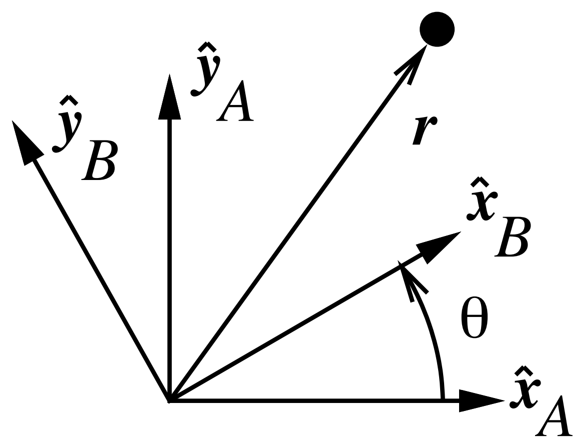
Translation



$$\vec{r}_A = \vec{r}_B + \vec{t}_A$$

Spatial Relationships

Rotations



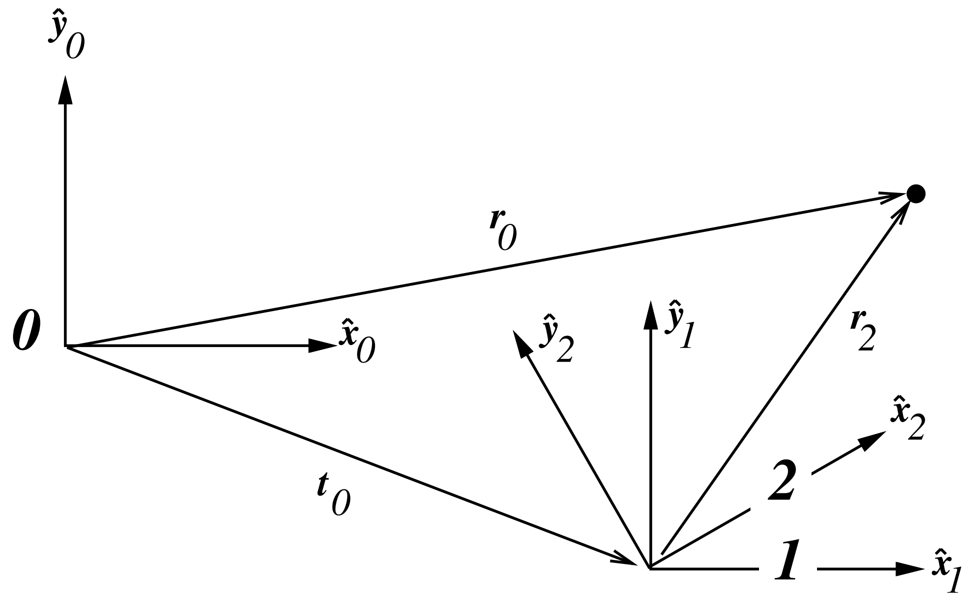
$$\vec{r}_A \;=\; {}_A\mathbf{R}_B \; \vec{r}_B$$

$$\begin{bmatrix} rx_A \\ ry_A \\ rz_A \end{bmatrix} = \begin{bmatrix} \hat{i}_A \cdot \hat{i}_B & \hat{i}_A \cdot \hat{j}_B & \hat{i}_A \cdot \hat{k}_B \\ \hat{j}_A \cdot \hat{i}_B & \hat{j}_A \cdot \hat{j}_B & \hat{j}_A \cdot \hat{k}_B \\ \hat{k}_A \cdot \hat{i}_B & \hat{k}_A \cdot \hat{j}_B & \hat{k}_A \cdot \hat{k}_B \end{bmatrix} \begin{bmatrix} rx_B \\ ry_B \\ rz_B \end{bmatrix}$$

where $\hat{i}, \hat{j}, \hat{k}$ represent the basis vectors for a coordinate frame.

Spatial Relationships

The Homogeneous Transform



$$ {}_0\mathbf{T}_2 = \left[\begin{array}{c|c} {}_1\mathbf{R}_2 & \vec{t}_0 \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \in SE(3) \qquad \vec{r}_2 = \begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix}_2 $$

Now,

$$ \begin{aligned} \vec{r}_0 &= {}_0\mathbf{T}_2 \vec{r}_2 \\ &= {}_1\mathbf{R}_2 \vec{r}_2 + \vec{t}_0 \end{aligned} $$

The Homogeneous Transform

$$
{}_AT_B = \left[\begin{array}{c|c} {}_AR_B & \vec{t}_B^{\,A} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right] \qquad\qquad \vec{r}_A = {}_AT_B\ \vec{r}_B
$$

$$
trans(\vec{t}) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
rot(\bar{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) & 0 \\ 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
rot(\bar{y}, \theta) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta) & 0 & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
rot(\bar{z}, \theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

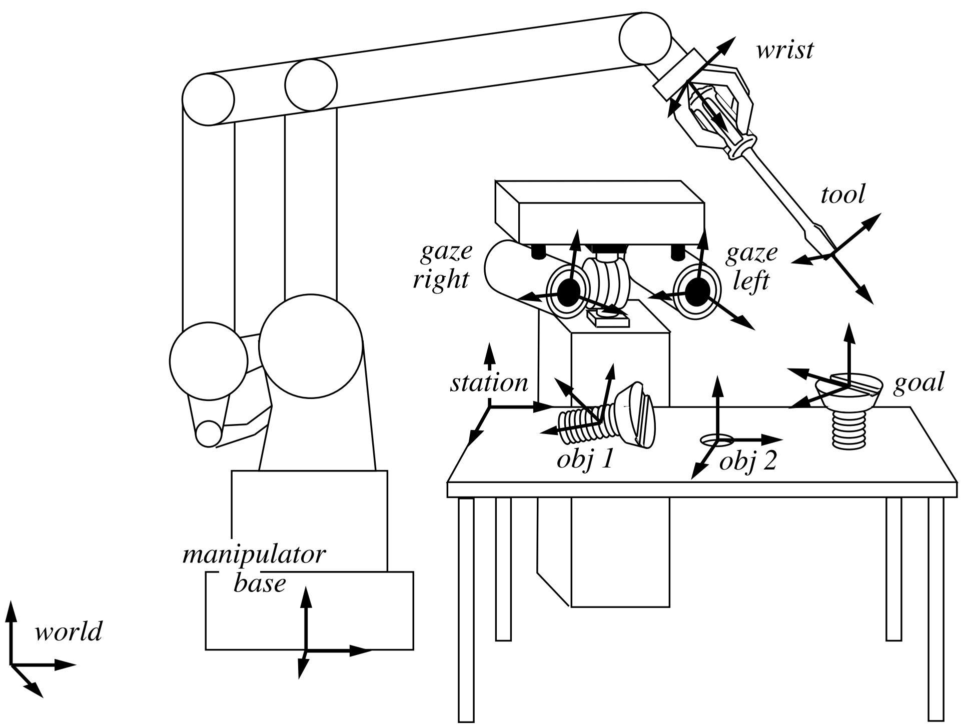
Spatial Relationships

Inverting the Homogeneous Transform

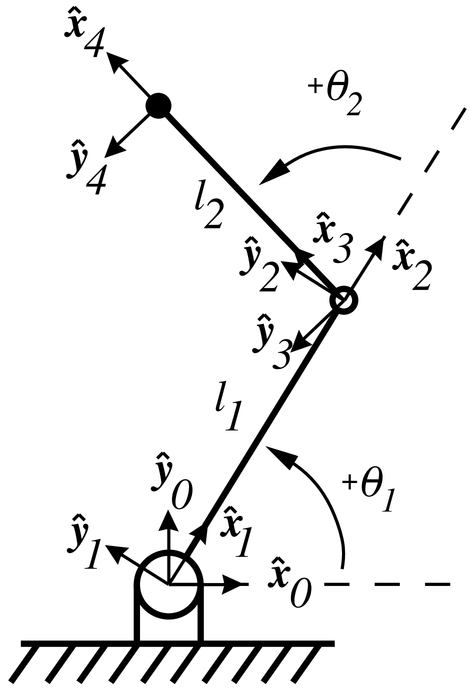
$$
{}_{A}\mathbf{T}_{B} = \begin{bmatrix} \hat{x}_B^A & \hat{y}_B^A & \hat{z}_B^A & \vec{t} \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}_{B}\mathbf{T}_{A} = [\ {}_{A}\mathbf{T}_{B}]^{-1} = \begin{bmatrix} & (\hat{x}_B^A)^T & & (-\vec{t} \cdot \hat{x}_B^A) \\ & (\hat{y}_B^A)^T & & (-\vec{t} \cdot \hat{y}_B^A) \\ & (\hat{z}_B^A)^T & & (-\vec{t} \cdot \hat{z}_B^A) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Forward Kinematics: $\Theta \mapsto X$



Forward Kinematics: EXAMPLE



$$
{}_0T_3 = \begin{bmatrix} cos_{12} & -sin_{12} & 0 & l_1 cos_1 + l_2 cos_{12} \\ sin_{12} & cos_{12} & 0 & l_1 sin_1 + l_2 sin_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

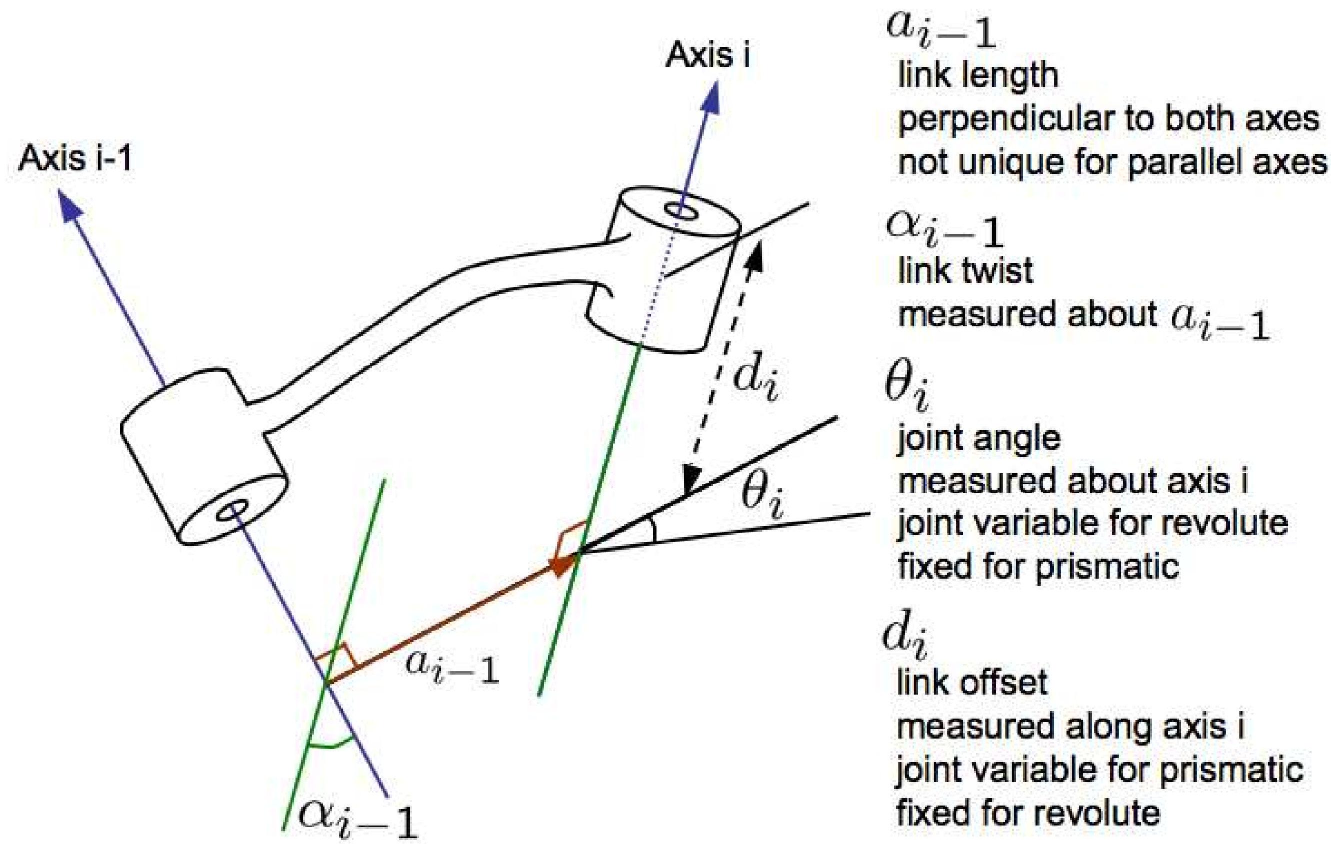
$$
\begin{aligned}
x &= l_1 cos(\theta_1) + l_2 cos(\theta_1 + \theta_2) \\
y &= l_1 sin(\theta_1) + l_2 sin(\theta_1 + \theta_2)) \\
\theta &= \theta_1 + \theta_2
\end{aligned}
$$

Useful Trignometric Identities

$$
\begin{aligned}
cos(\theta_1 + \theta_2) &= cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) \\
cos(\theta_1 - \theta_2) &= cos(\theta_1)cos(\theta_2) + sin(\theta_1)sin(\theta_2) \\
sin(\theta_1 + \theta_2) &= sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2) \\
sin(\theta_1 - \theta_2) &= sin(\theta_1)cos(\theta_2) - cos(\theta_1)sin(\theta_2) \\
sin(\theta_2) &= cos(\theta_1)sin(\theta_1 + \theta_2) - sin(\theta_1)cos(\theta_1 + \theta_2) \\
cos(\theta_2) &= cos(\theta_1)cos(\theta_1 + \theta_2) + sin(\theta_1)sin(\theta_1 + \theta_2)
\end{aligned}
$$

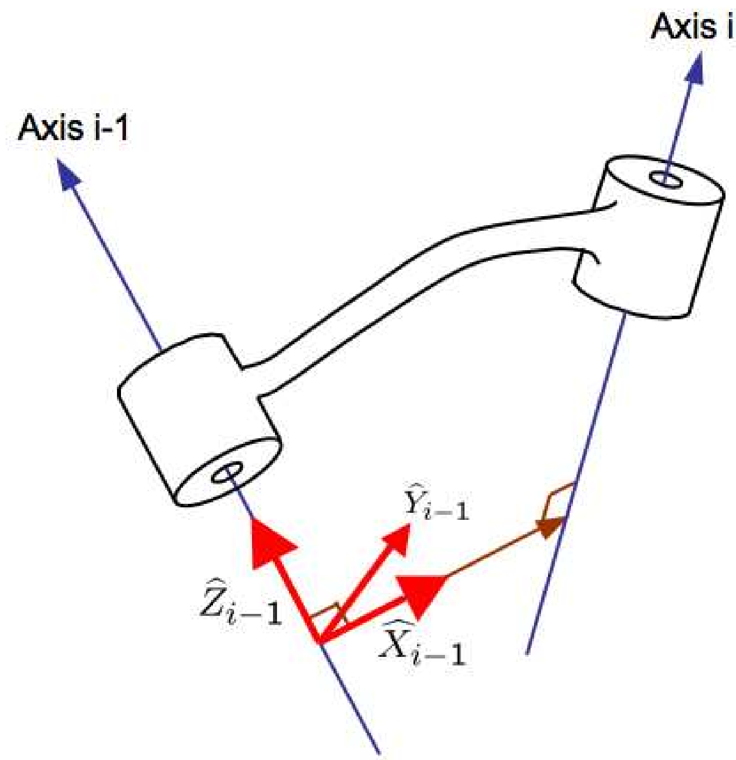
Denavit-Hartenberg Parameters

from $frame_{i-1}$ to $frame_i$:



3 fixed parameters per joint, one variable

Denavit-Hartenberg Coordinate Frames



$\widehat{Z}_{i-1}$

along joint axis of joint **i-1**

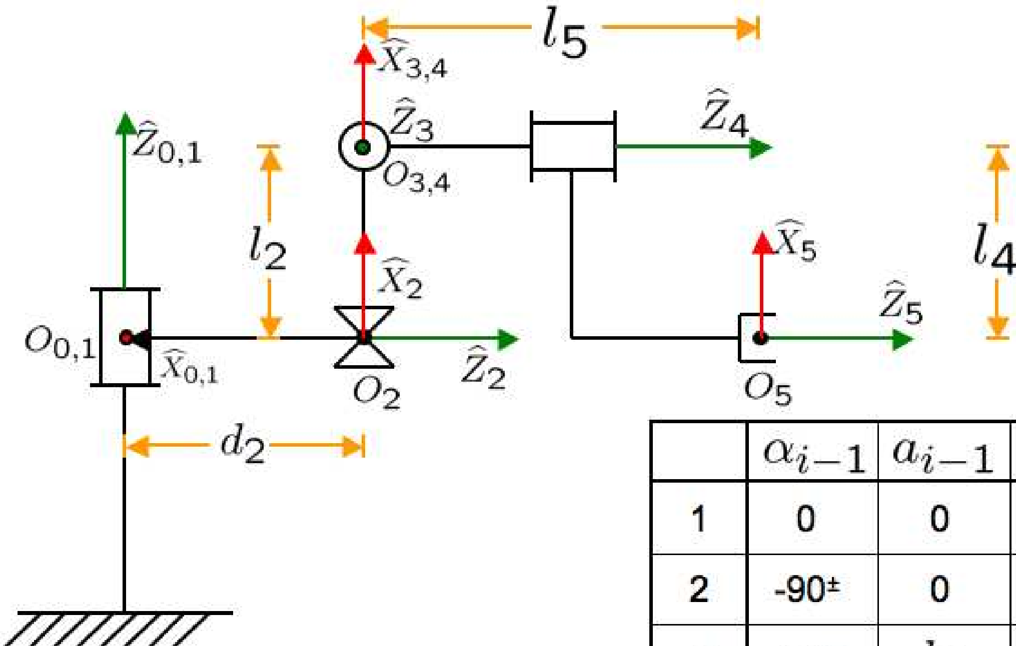
$\widehat{X}_{i-1}$

along perpendicular from joint axis **i-1** to joint axis **i** (note special case for intersecting axes)

$\widehat{Y}_{i-1}$

results from right-hand-rule

Denavit-Hartenberg - 4 DOF example



	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90±	0	d_2	-90±
3	-90±	l_2	0	θ_3
4	90±	0	0	θ_4
5	0	$-l_4$	l_5	0

Denavit-Hartenberg - Parametric Homogeneous Transform

$$
\begin{aligned}
\alpha_i &= \text{ the angle between } \hat{Z}_i \text{ and } \hat{Z}_{i+1} \text{ measured about } \widehat{X}_i \\
a_i &= \text{ the distance from } \hat{Z}_i \text{ to } \hat{Z}_{i+1} \text{ measured along } \widehat{X}_i \\
d_i &= \text{ the distance from } \widehat{X}_{i-1} \text{ to } \widehat{X}_i \text{ measured along } \hat{Z}_i \\
\theta_i &= \text{ the angle between } \widehat{X}_{i-1} \text{ and } \widehat{X}_i \text{ measured about } \hat{Z}_i
\end{aligned}
$$

$$
\begin{aligned}
{}_{i\text{-}1}T_i &= R_X(\alpha_{i-1})\, D_X(a_{i-1})\, R_Z(\theta_i)\, D_Z(d_i) \\
&= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$