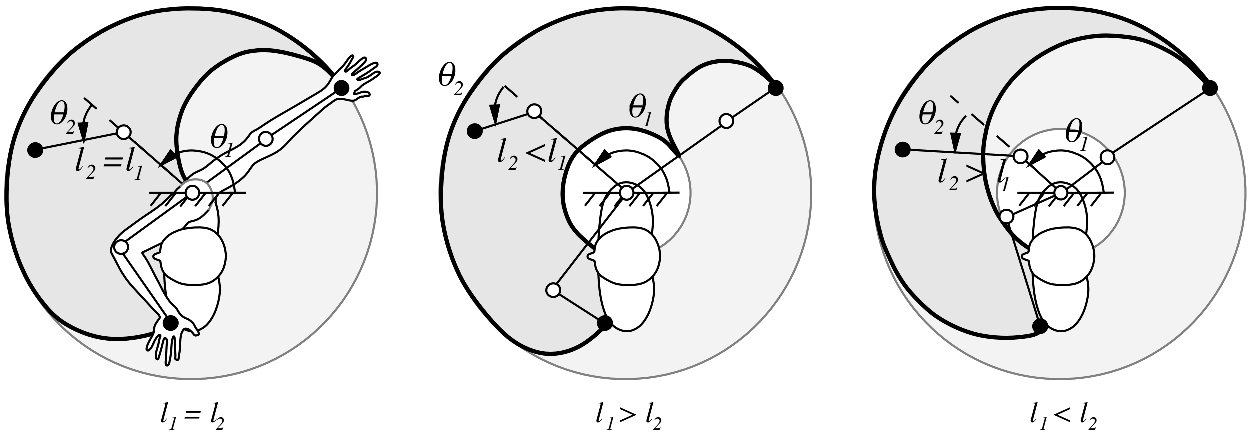
Inverse Kinematics: $X \mapsto \Theta$

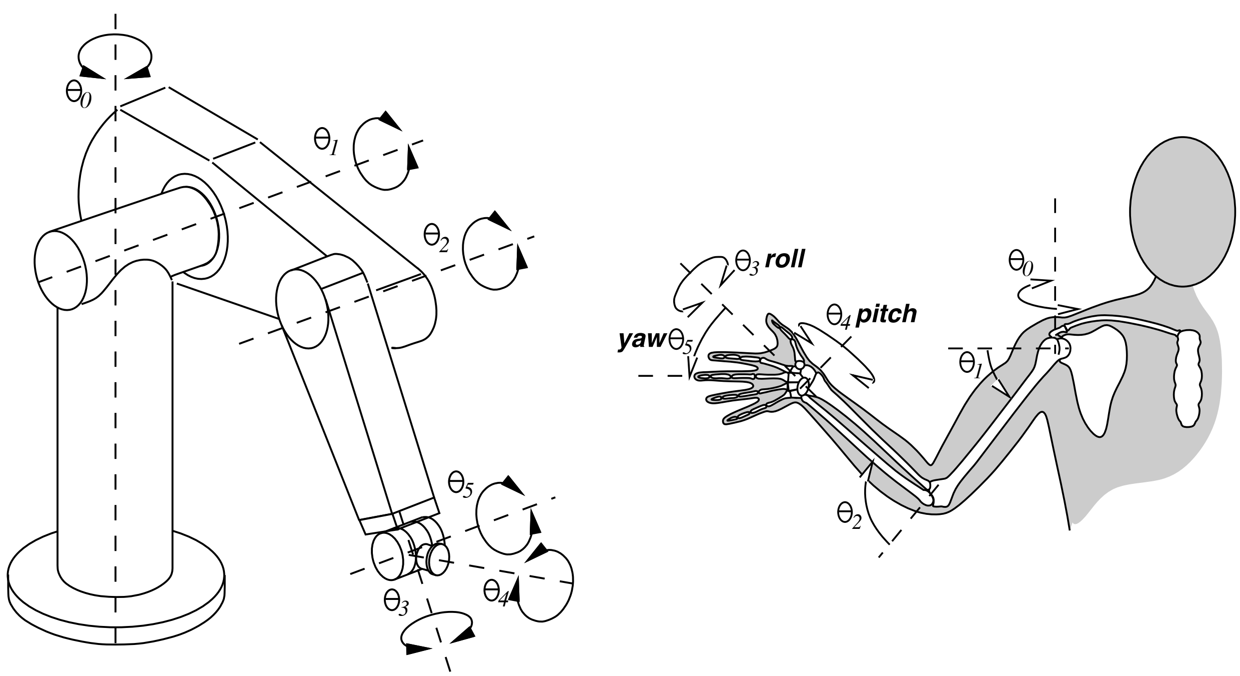
reachability, dexterity, multiple solutions



Closed-Form Inverse Kinematic Solutions

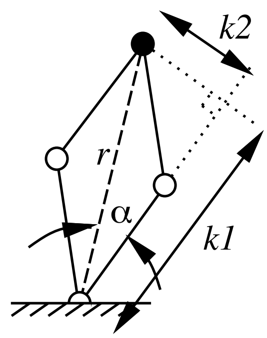
- Pieper (ca. 1968) general inverse kinematic solution

  6 revolute joints have a closed form solution if 3 neighboring joint axes intersect at a point



- Paul (ca. 1981) homogeneous transform-based generalized IK
- Geometric Techniques

Inverse Kinematics: EXAMPLE



- eliminate θ_1, solve for two unique θ_2 solutions:

$$\begin{aligned} r^2 &= x^2 + y^2, \; and \\ x &= l_1 c_1 + l_2 c_{12} \\ y &= l_1 s_1 + l_2 s_{12} \end{aligned}$$

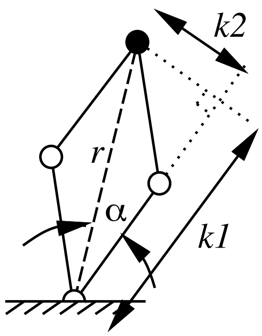
$$\begin{aligned} s_{12} &= s_1 c_2 + c_1 s_2 \\ c_{12} &= c_1 c_2 - s_1 s_2 \end{aligned}$$

$$\begin{aligned} r^2 = x^2 + y^2 &= l_1^2 c_1^2 + 2 l_1 l_2 c_1 c_{12} + l_2^2 c_{12}^2 \\ &\quad + l_1^2 s_1^2 + 2 l_1 l_2 s_1 s_{12} + l_2^2 s_{12}^2 \\ &= l_1^2 + 2 l_1 l_2 c_2 + l_2^2 \end{aligned}$$

and,

$$c_2 = \frac{r^2 - l_1^2 - l_2^2}{2 l_1 l_2}, \; c_2 \in [-1, +1]$$

Inverse Kinematics: EXAMPLE



$$k_1 = rc_\alpha = l_1 + l_2c_2$$
$$k_2^{+/-} = rs_\alpha = l_2s_2^{+/-}$$

- solve for both θ_2 solutions

$$\begin{aligned} s_2^2 + c_2^2 &= 1 \\ s_2^2 &= 1 - c_2^2 \\ s_2^{+/-} &= +/-(1 - c_2^2)^{1/2} \end{aligned}$$

$$\theta_2^{+/-} = tan^{-1}\frac{s_2^{+/-}}{c_2}$$

and,

$$\alpha^{+/-} = tan^{-1}\frac{k_2^{+/-}}{k_1}$$

Therefore,

$$\begin{aligned} x = k_1c_1 + k_2s_1 &= (rcos_\alpha)c_1 + (rsin_\alpha)s_1 \\ &= rcos(\alpha + \theta_1) \\ y = k_1s_1 + k_2c_1 &= (rcos_\alpha)s_1 + (rsin_\alpha)c_1 \\ &= rsin(\alpha + \theta_1) \end{aligned}$$

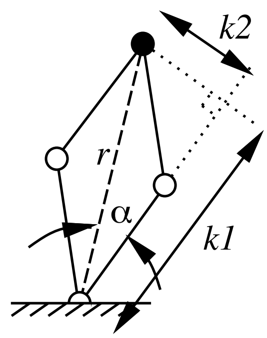
and

$$tan(\alpha + \theta_1) = \frac{rsin(\alpha + \theta_1)}{rcos(\alpha + \theta_1)} = \frac{y}{x}$$

so that,

$$\theta_1^{+/-} = tan^{-1}\frac{y}{x} - \alpha^{+/-}.$$

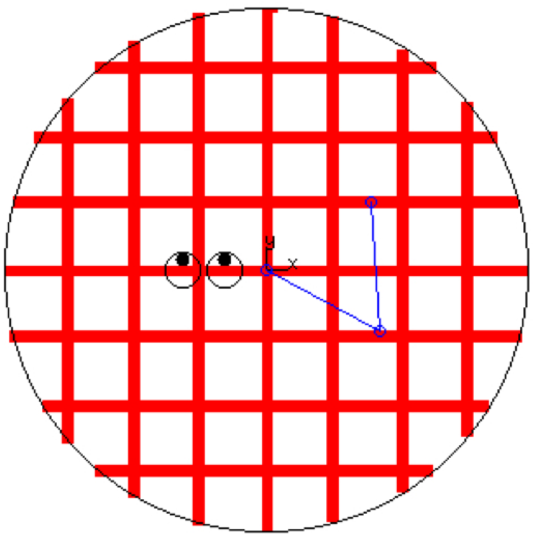
Inverse Kinematics: EXAMPLE

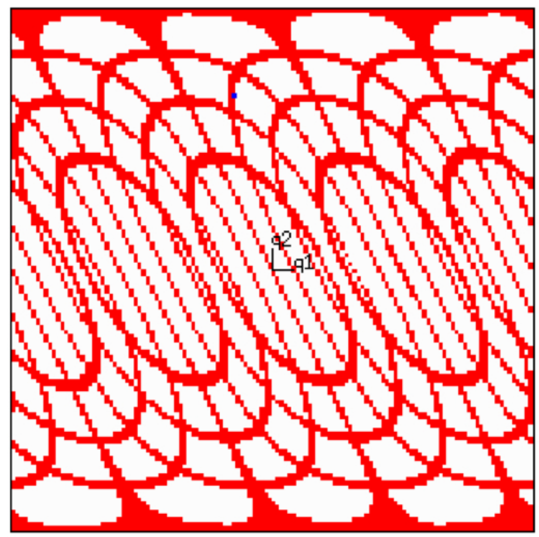


GIVEN (x,y) endpoint position goal:

$$
\begin{aligned}
& r^2 = x^2 + y^2 \\
& c_2 = (r^2 - l_1^2 - l_2^2)/(2l_1l_2) \\
& \text{if } (-1 \leq c_2 \leq +1) \\
& \qquad s_2^{+/-} = +/-(1 - c_2^2)^{1/2} \\
& \qquad \theta_2^{+/-} = tan^{-1}\frac{s_2^{+/-}}{c_2} \\
& \qquad k_1 = l_1 + l_2c_2 \\
& \qquad k_2^{+/-} = l_2s_2^{+/-} \\
& \qquad \alpha^{+/-} = tan^{-1}\frac{k_2^{+/-}}{k1} \\
& \qquad \theta_1^{+/-} = tan^{-1}\frac{y}{x} - \alpha^{+/-} \\
& \text{else “out of reach”}
\end{aligned}
$$

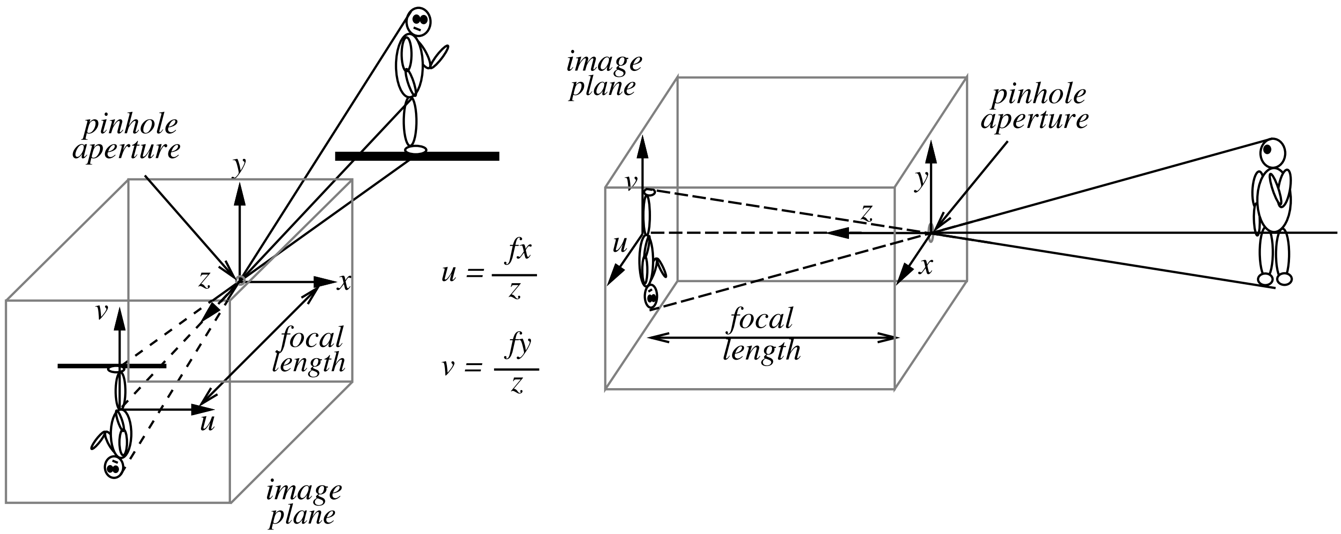
Inverse Kinematics: EXAMPLE





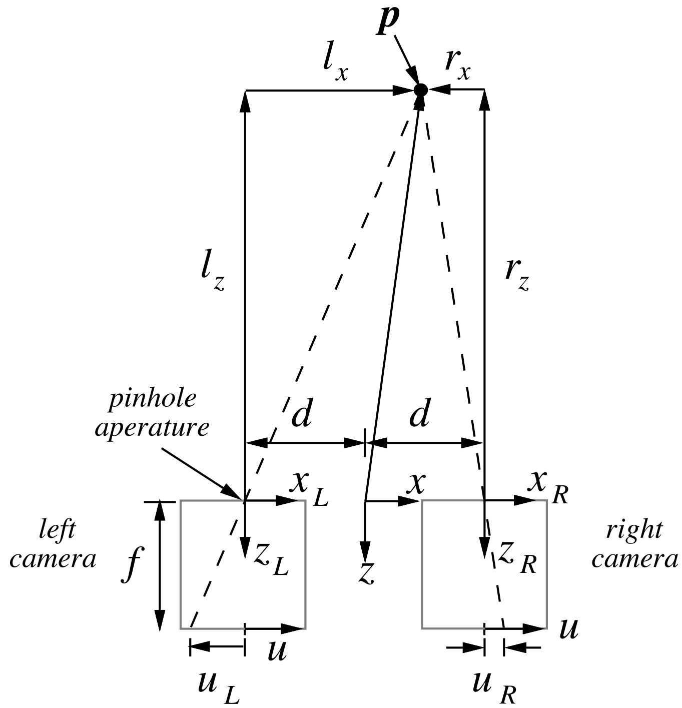
Pinhole Camera - Perspective Distortion

...another kinematic system



- **vanishing point** - point at which parallel lines in three space meet on the image plane due to perspective distortion.
- **orthographic projection** - as range goes to infinity, the geometric distortion due to variations in depth goes to zero.
- **shallow structure** - images of objects whose depth is small compared to their range are approximately orthographic.

Depth Encoded as Disparity



Consider this simple binocular configuration. In this geometry, the stereo system encodes depth entirely in terms of disparity. Under these conditions

$$u_L = \frac{f(x-d)}{z} \qquad\qquad u_R = \frac{f(x+d)}{z}$$

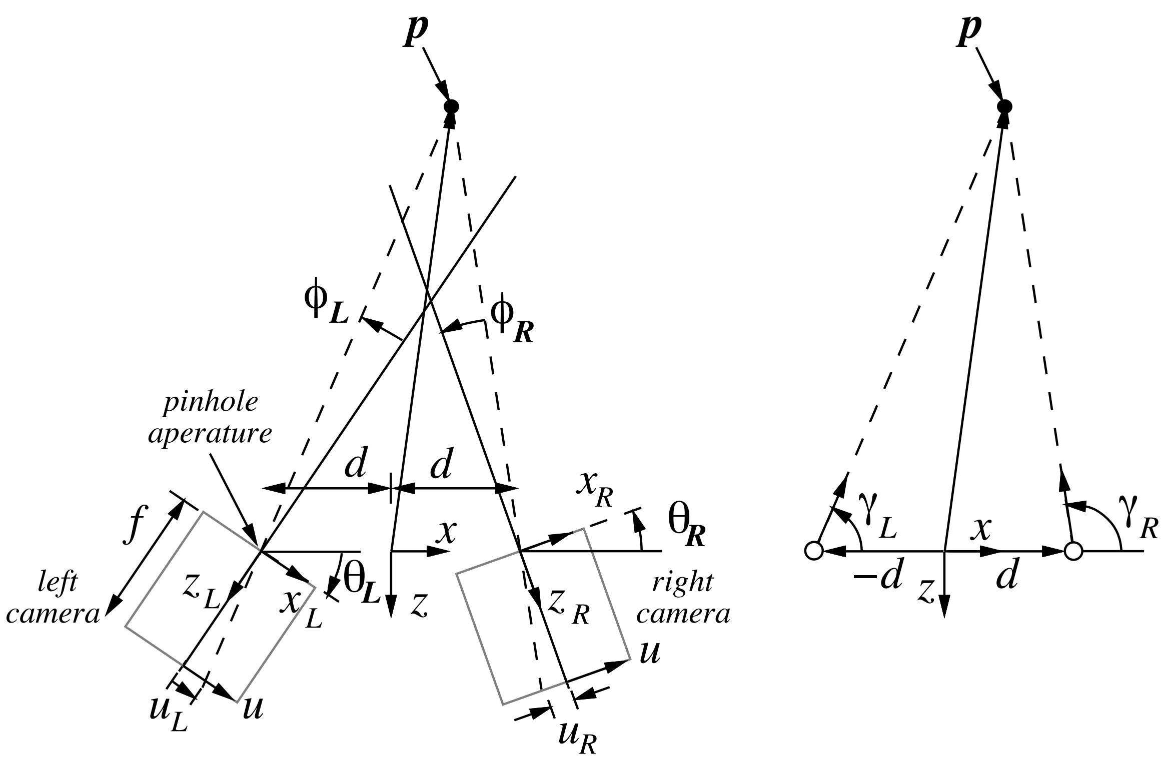
$$\begin{aligned} zu_R &= f(x+d) \\ zu_L &= f(x-d) \end{aligned}$$

$$z(u_R - u_L) = 2df$$

So by eliminating x , we may solve directly for z

$$z = \frac{2df}{(u_R - u_L)}$$

Recovering Space - Binocular Stereopsis



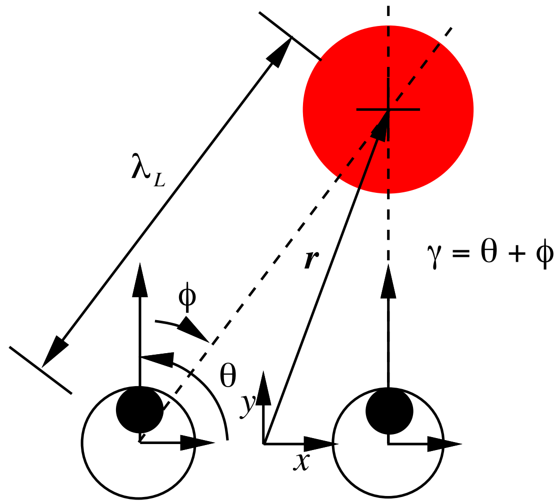
$$\lambda_L cos(\gamma_L) - d = \lambda_R cos(\gamma_R) + d$$
$$\lambda_L sin(\gamma_L) = \lambda_R sin(\gamma_R),$$

$$\lambda_L = \frac{2d sin(\gamma_R)}{sin(\gamma_R - \gamma_L)}$$
$$\lambda_R = \frac{2d sin(\gamma_L)}{sin(\gamma_R - \gamma_L)}$$

Depth by **vergence** and **disparity**.

Stereo Conditioning - Localizability

Careful! Roger's coordinate frame is slightly different!



...so for Roger, we get...

$$\begin{aligned} x &= -d + \lambda_L cos(\gamma_L) \\ &= d + \lambda_R cos(\gamma_R) \\ &= f_x(\gamma_L, \gamma_R) \end{aligned}$$

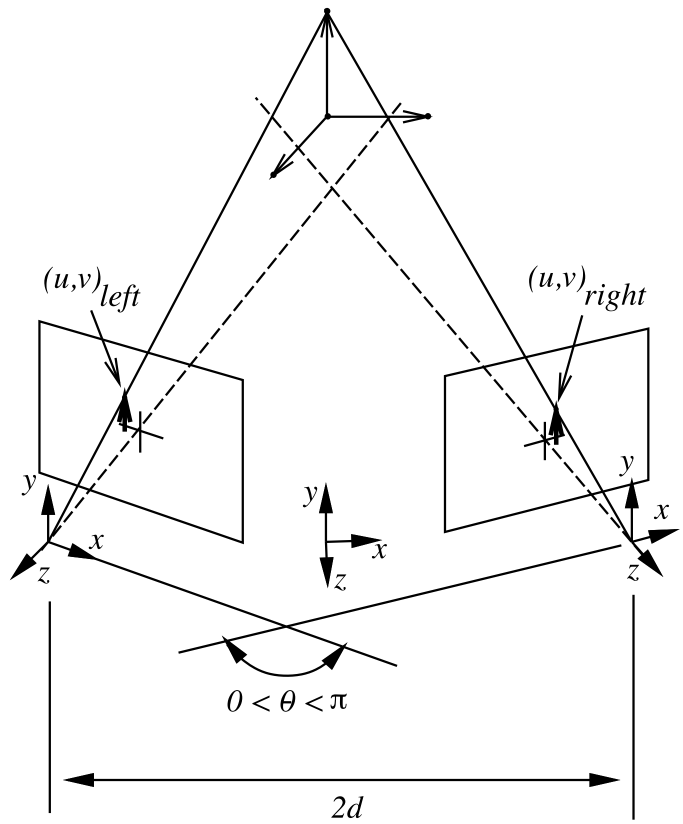
$$\begin{aligned} y &= \lambda_L sin(\gamma_L) \\ &= \lambda_R sin(\gamma_R) \\ &= f_y(\gamma_L, \gamma_R) \end{aligned}$$

where:

$$\lambda_L = \frac{2d sin(\gamma_R)}{sin(\gamma_R - \gamma_L)}$$

$$\lambda_R = \frac{2d sin(\gamma_L)}{sin(\gamma_R - \gamma_L)}$$

Weak Perspective Affine Transformations



$$\begin{bmatrix} u_L \\ v_L \\ u_R \\ v_R \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Weak Perspective - cont.

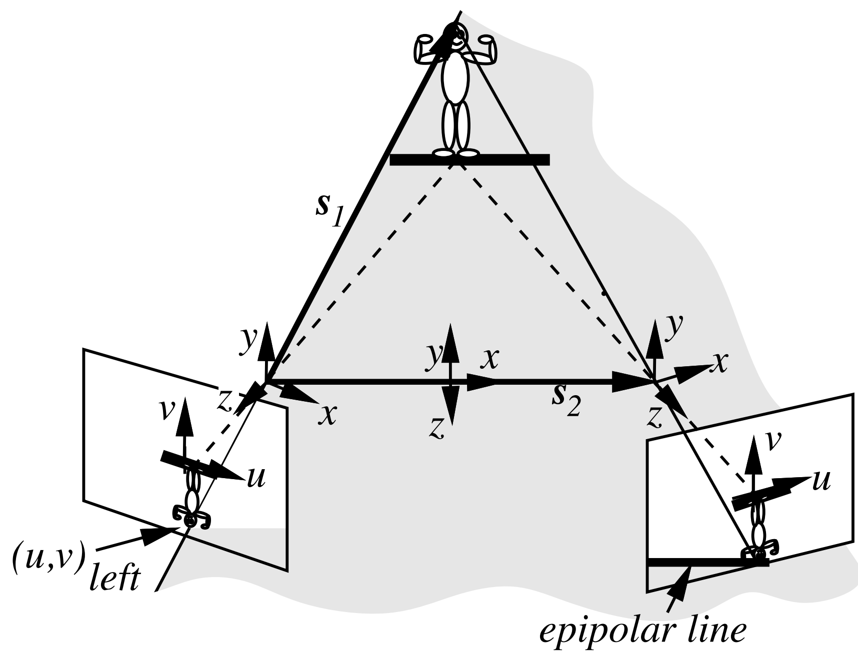
With 4 non-coplanar points, this leads to
16 equations in 16 unknowns

or since;

$$\begin{bmatrix} u_L^0 \\ u_L^1 \\ u_L^2 \\ u_L^3 \end{bmatrix} = \begin{bmatrix} A_{00}x^0 & A_{01}y^0 & A_{02}z^0 & A_{03} \\ A_{00}x^1 & A_{01}y^1 & A_{02}z^1 & A_{03} \\ A_{00}x^2 & A_{01}y^2 & A_{02}z^2 & A_{03} \\ A_{00}x^3 & A_{01}y^3 & A_{02}z^3 & A_{03} \end{bmatrix}$$

the solution can be obtained from $4 \times$ (4 *equations in* 4 *unknowns*)

The Epipolar Constraint



Epipolar plane: defined by the line joining the focal points of the stereo system and the ray from the left focal point to the image feature of interest.

Epipolar line: The intersection of the epipolar plane on the left and right image planes.

The Epipolar Constraint - continued

Alternatively, the inverse of the Amatrix:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} A_{00}^{-1} & A_{01}^{-1} & A_{02}^{-1} & A_{03}^{-1} \\ A_{10}^{-1} & A_{11}^{-1} & A_{12}^{-1} & A_{13}^{-1} \\ A_{20}^{-1} & A_{21}^{-1} & A_{22}^{-1} & A_{23}^{-1} \\ A_{30}^{-1} & A_{31}^{-1} & A_{32}^{-1} & A_{33}^{-1} \end{bmatrix} \begin{bmatrix} u_L \\ v_L \\ u_R \\ v_R \end{bmatrix}$$

the forth column yields:

$$1 = A_{30}^{-1} u_L + A_{31}^{-1} v_L + A_{32}^{-1} u_R + A_{33}^{-1} v_R$$

Summary - Hand-Eye Spatial Transformations