Basic Tools of Linear Control Theory

Outline

- Spinal Motor Units
- Negative Feedback
- Open- and Closed-Loop Control
- The Spring-Mass-Damper
- Lyapunov Stability
- Laplace Transform
- the Characteristic Equation
- Equilibrium Setpoint Control - A Robot Controller
  class exercise - Roger’s eye and PD control
- Frequency-Domain Response Demonstration
  Roger’s eye frequency-domain response
Motor Circuits

- α-motor neurons initiate motion—they’re fast
- each will innervate an average of 200 muscle fibers.
- relatively slow γ-motor neuron regulates muscle tone by setting the reference length of the spindle receptor.
- Golgi tendon organ measures the tension in the tendon and inhibits the α-motor neuron if it exceeds safe levels
Negative Feedback

• If (spindle length > reference), the $\alpha$-motor neuron cause a contraction of the muscle tissue

• if (spindle length < reference), the $\alpha$-motor neuron is inhibited, allowing the muscle to extend

Negative Feedback

...the $\alpha$-motor neuron changes its output so as to cancel some of its input...
Negative Feedback

• first submitted for a patent in 1928 by Harold S. Black
• it explained the operating principle of many devices including Watt’s governor that pre-dated it by some 40 years.
• catalyzed the field of cybernetics
• now heralded as the fundamental principle of stability in compensated dynamical systems

The Muscle Stretch Reflex
Open- and Closed-Loop Control

open-loop -
a trigger event causes a response without further stimulation

withdrawl reflex

closed-loop -
a (time-varying) setpoint is achieved by constantly measuring and correcting in order to actively reject disturbances

Norbert Weiner - cybernetics (helmsman), homeostasis, endocrine system
The Spring-Mass-Damper

\[ F_k = -Kx \]
\[ F_b = -Bv = -B\dot{x} \]

\[ \sum F = m\ddot{x} = f(t) - B\dot{x} - Kx \]

\[ m\ddot{x} + B\dot{x} + Kx = f(t), \quad \text{or} \]
\[ \ddot{x} + \left(\frac{B}{m}\right)\dot{x} + \left(\frac{K}{m}\right)x = f(t)/m = f(t) \quad \text{“specific” applied force} \]

\[ \ddot{x} + \left(\frac{B}{m}\right)\dot{x} + \left(\frac{K}{m}\right)x = 0 \quad \text{the characteristic equation} \]

arbitrary references require a change of variables:

\[ x'(t) = x(t) - x_{ref} \]
\[ \dot{x}' = \dot{x} \]
The Spring-Mass-Damper

\[ \ddot{x} + \frac{B}{m} \dot{x} + \frac{K}{m} x = 0 \]

we can write this another way:

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 0 \]

*harmonic oscillator*

where:

\[ \omega_n = \sqrt{\frac{K}{m}} \quad [\text{rad/sec}] - \text{natural frequency} \]

\[ \zeta = \frac{B}{2\sqrt{Km}} \quad 0 \leq \zeta \leq \infty - \text{damping ratio} \]
Closed-Loop Control

sample and hold $\Delta t = \tau$ where $\frac{1}{\tau}$ [Hz] is the servo rate

$\tau_{\text{motor}}$ digital control

$\tau_{\text{motor}}$ digital control

$\Delta t \rightarrow 0$
Roger MotorUnits.c
Master Control Procedure

/* == the simulator executes control_roger() once ==*/
/* == every simulated 0.001 second (1000 Hz) ==*/
class_roger(roger, time)
Robot * roger;
double time;
{
    update_setpoints(roger);

    // turn setpoint references into torques
    PDController_base(roger, time);
    PDController_arms(roger, time);
    PDController_eyes(roger, time);
}

Copyright ©2019 Roderic Grupen
double Kp_eye, Kd_eye;
// gain values set in enter_params()

/* Eyes PD controller:
   */
/* -pi/2 < eyes_setpoint < pi/2 for each eye */
PDController_eyes(roger, time)
Robot * roger;
double time;
{
    int i;
    double theta_error;

    for (i = 0; i < NEYES; i++) {
        theta_error = roger->eyes_setpoint[i]
                      - roger->eye_theta[i];
        // roger->eye_torque[i] = ...
    }
}
Roger MotorUnits.c
PDcontroller_arms()

double Kp_arm, Kd_arm;
// gain values set in enter_params()

/* Arms PD controller: -pi < arm_setpoint < pi */
/* for the shoulder and elbow of each arm */
PDController_arms(roger, time)
Robot * roger;
double time;
{
    int i;
    double theta_error;

    for (i = LEFT; i <= RIGHT; ++i) {
        theta_error = roger->arm_setpoint[i][0]
                      - roger->arm_theta[i][0];

        // -M_PI < theta_error < +M_PI

        // roger->arm_torque[i][0] = ...
        // roger->arm_torque[i][1] = ...
    }
}
Analytic Stability —
Lyapunov’s Second/Direct Method

**Stability** - the origin of the state space is stable if there exists a region, $S(r)$, such that states which start within $S(r)$ remain within $S(r)$.

**Asymptotic Stability** - a system is asymptotically stable in $S(r)$ if as $t \to \infty$, the system state approaches the origin of the state space.
Define: an arbitrary scalar function, $V(x, t)$, called a Lyapunov function, continuous is all first derivatives, where $x$ is the state and $t$ is time,

Iff: If the function, $V(x, t)$, exists such that:

(a) $V(0, t) = 0$, and
(b) $V(x, t) > 0$, for $x \neq 0$ (positive definite), and
(c) $\partial V/\partial t < 0$ (negative definite),

Then: the system described by $V$ is asymptotically stable in the neighborhood of the origin.

...if a system is stable, then there exists a suitable Lyapunov function.

...if, however, a particular Lyapunov function does not satisfy these criteria, it is not necessarily true that this system is unstable.
EXAMPLE: spring-mass-damper

system dynamics:
\[ \ddot{x} + \frac{B}{m} \dot{x} + \frac{K}{m} x = 0 \]

\[ x = 0 \]

\[ E = \int_0^v (mv) dv + \int_0^x (Kx) dx \]
\[ = \frac{1}{2}mv^2 + \frac{1}{2}Kx^2 \]
\[ = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2 \]

Lyapunov function:

\[ V(x, t) = E = \frac{m\dot{x}^2}{2} + \frac{Kx^2}{2} \]

\(a\) \( V(0, t) = 0 \), \( \sqrt{\ } \)

\(b\) \( V(x, t) > 0 \), \( \sqrt{\ } \)

\(c\) \( \partial V/\partial t \) negative definite?
EXAMPLE: spring-mass-damper

Lyapunov function:

\[ V(x, t) = E = \frac{m\dot{x}^2}{2} + \frac{Kx^2}{2} \]

\[
\frac{dE}{dt} = m\ddot{x}\dot{x} + Kx\dot{x}
\]

\[
\frac{dE}{dt} = m\dot{x}[-(B/m)\dot{x} - (K/m)x] + Kx\dot{x}
\]

\[
\frac{dE}{dt} = -B\dot{x}^2
\]

stable? or not stable?
EXAMPLE: spring-mass-damper

...the entire state space is asymptotically stable for $B > 0$. 
Recap: Introduction to Control

So far, we have:

• introduced the concept of negative feedback in robotics and biology;
• proposed the spring-mass-damper (SMD) as a prototype for proportional-derivative (PD) control;
• we derived the dynamics for the SMD using Newton’s laws and a free body diagram; and
• we introduced Lypunov’s Direct Method to show the SMD (and thus PD control) is asymptotically stable.

Now: we describe more tools for analyzing closed-loop linear controllers — the Laplace transform and transfer functions
Tools: Complex Numbers

Cartesian form: \( s = \sigma + j\omega \)

- \( \sigma = Re(s) \) is the real part of \( s \)
- \( \omega = Im(s) \) is the imaginary part of \( s \)
- \( j = \sqrt{-1} \) (sometimes I may use \( i \))

Polar form: \( s = re^{j\phi} \)

- \( r = \sqrt{\sigma^2 + \omega^2} \) is the modulus or magnitude of \( s \)
- \( \phi = \text{atan}(\omega/\sigma) \) is the angle or phase of \( s \)

- Euler’s formula: \( e^{j\phi} = \cos(\phi) + j\sin(\phi) \)

Therefore, complex exponential of \( s = \sigma + j\omega \):

\[
e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t} [\cos(\omega t) + j\sin(\omega t)]
\]
Laplace Transform

...so what does this do for us?

if we assume that the robot movements are functions of time $f(t)$, such that

$$f(t) \sim e^{st}$$

then, from calculus:

$$\frac{d}{dt} [f(t)] = \dot{f}(t) \sim se^{st}$$

$$\int f(t)dt \sim \frac{1}{s}e^{st}$$

let’s say this a different way (ignoring some details about boundary conditions for now), if $\mathcal{L} [f(t)] = F(s)$, then

$$\mathcal{L} \left[ \frac{df}{dt} \right] = sF(s) , \text{ and}$$

$$\mathcal{L} \left[ \int f(t) dt \right] = \frac{1}{s}F(s)$$
for example, \[ \dot{f} + af = 0 \]
i.e. the “slope” of function \( f \) \((df/dt)\) is proportional to the value of the function, \( df/dt = -af \)

assuming \( f(t) \sim e^{st} \):

\[
\begin{align*}
    sF(s) + aF(s) &= 0 \\
    (s + a)F(s) &= 0
\end{align*}
\]

and the first-order differential equation is transformed into polynomial \((s + a)\),

root \((s = -a)\) tells us more about function \( f(t) \),

\[
f(t) \sim A_0 + A_1 e^{-at}
\]

where coefficients \( A_0 \) and \( A_1 \) are constants that depend on boundary conditions, i.e. if \( f(0) = 1 \) and \( f(\infty) = 0 \) verify that: \( A_0 = 0 \) and \( A_1 = 1 \)
The Harmonic Oscillator

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \quad \xrightarrow{\mathcal{L}(\cdot)} \quad \left[ s^2 + 2\zeta\omega_n s + \omega_n^2 \right] X(s) = 0 \]

yields the *characteristic equation* of the 2nd-order oscillator in the complex frequency domain

\[ s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \]
Roots of the Characteristic Equation

\[ s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \]

roots ⇒ values of \( s \) in \( Ae^{st} \) that satisfy the original differential equation

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 0 \]

\[ s_{1,2} = \frac{-2\zeta \omega_n \pm \sqrt{(2\zeta \omega_n)^2 - 4\omega_n^2}}{2} = \frac{2\omega_n[-\zeta \pm \sqrt{\zeta^2 - 1}]}{2} \]

\[ = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}, \]

three cases:

- repeated real roots (\( \zeta = 1 \))
- distinct real roots (\( \zeta > 1 \))
- complex conjugates roots (\( \zeta < 1 \))
Roots of the Characteristic Equation

For two distinct roots

\[ x(t) = A_0 + A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

the solution in \( t \in [0, \infty) \) requires three boundary conditions to solve for three unknowns \( A_0, A_1, \) and \( A_2 \)

\[
\begin{align*}
  x(0) &= x_0 = A_0 + A_1 + A_2 \\
  \dot{x}(0) &= \dot{x}_0 = s_1 A_1 + s_2 A_2, \\
  x(\infty) &= x_\infty = A_0
\end{align*}
\]

so, a complete time-domain solution is determined

\[
x(t) = x_\infty + \frac{(x_0 - x_\infty) s_2 - \dot{x}_0}{s_2 - s_1} e^{s_1 t} + \frac{(x_0 - x_\infty) s_1 - \dot{x}_0}{s_1 - s_2} e^{s_2 t}
\]
Roots of the Characteristic Equation

given boundary conditions $x_0 = \dot{x}_0 = 0$ and $x_\infty = 1.0$ the solution simplifies to

$$x(t) = 1.0 - \frac{s_2}{s_2 - s_1} e^{s_1 t} - \frac{s_1}{s_1 - s_2} e^{s_2 t}$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{plot.png}
\caption{\textit{\upshape Graph showing the behavior of $x(t)$ for different values of $\zeta$.}}
\end{figure}

\begin{itemize}
\item $\zeta = 0$
\item $\zeta = 0.1$
\item $\zeta = 0.2$
\item $\zeta = 0.4$
\item $\zeta = 0.7$
\item $\zeta = 1.0$
\item $\zeta = 2.0$
\end{itemize}

\end{itemize}

$$(K = 1.0 \, [N/m], \, M = 2.0 \, [kg])$$
Class Exercise

\[ a^2 + \frac{3}{2}a + \frac{9}{1} = 0 \]

\[ 10,000 < K < 100,000 \]

\[ K = 200.00 \quad \beta = 10.00 \]
Frequency-Domain Response

(a) $|C_{cltf}(s)|_{s=i\omega}$

$\zeta = 0.1, 0.2, 0.4, 1.0, 2.0$

 bandwidth:
 power ratio = $1/2$
 response = $1/\sqrt{2}$

(b) $|\phi_{cltf}|_{s=i\omega}$

$\zeta = 0.1, 0.2, 0.4, 2.0, 1.0$