Practical Stuff: Roger MotorUnits.c
Master Control Procedure

the simulator runs your control_roger() procedure inside "Project #1 - Basic Motor Units" every millisecond

```
/* == the simulator executes control_roger() once ==*/
/* == every simulated 0.001 second (1000 Hz)      ==*/
control_roger(roger, time)
Robot * roger;
double time;
{
  update_setpoints(roger);

  // turn setpoint references into torques
  PDController_base(roger, time);
  PDController_arms(roger, time);
  PDController_eyes(roger, time);
}
```

Roger MotorUnits.c
PDController_eyes()

```
double Kp_eye, Kd_eye;
// gain values set in enter_params()

/* Eyes PD controller:
/*    -pi/2 < eyes_setpoint < pi/2 for each eye */
PDController_eyes(roger, time)
Robot * roger;
double time;
{
  int i;
  double theta_error;

  for (i = 0; i < NEYES; i++) {
    theta_error = roger->eyes_setpoint[i]
                     - roger->eye_theta[i];
    // roger->eye_torque[i] = ...
  }
}
```

Roger MotorUnits.c PDcontroller_arms()

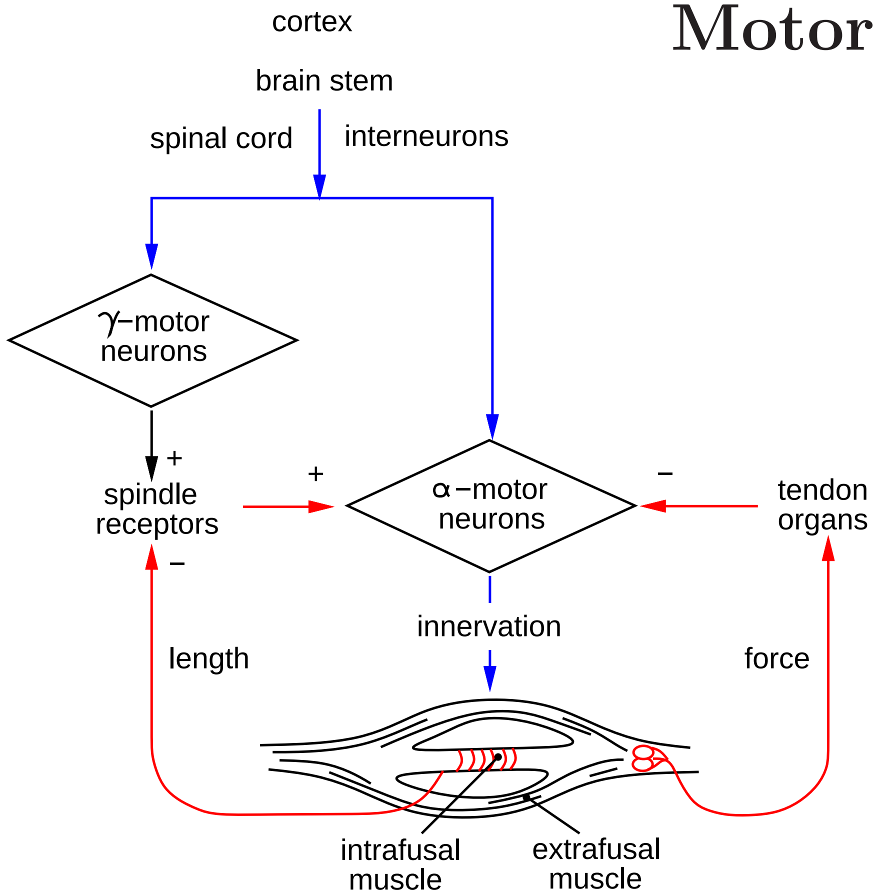
```
double Kp_arm, Kd_arm;
// gain values set in enter_params()

/* Arms PD controller: -pi < arm_setpoint < pi */
/* for the shoulder and elbow of each arm      */
PDController_arms(roger, time)
Robot * roger;
double time;
{
  int i;
  double theta_error;

  for (i = LEFT; i <= RIGHT; ++i) {
    theta_error = roger->arm_setpoint[i][0]
                     - roger->arm_theta[i][0];

    // -M_PI < theta_error < +M_PI

    // roger->arm_torque[i][0] = ...
    // roger->arm_torque[i][1] = ...
  }
}
```

Basic Tools of Linear Control Theory

Outline

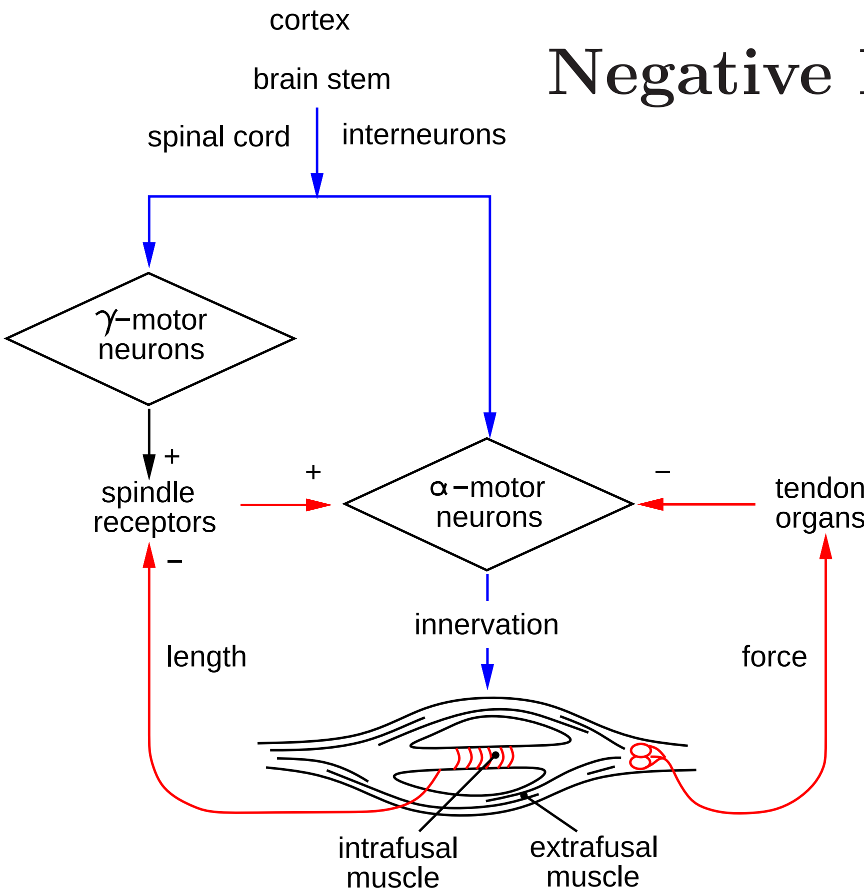
- Negative Feedback
- Open- and Closed-Loop Control
- The Canonical Spring-Mass-Damper - Lyapunov Stability
- Laplace Transform
- the Characteristic Equation
- Equilibrium Setpoint Control - A Robot Controller
  class exercise - Roger's eye and PD control
- Closed-Loop Transfer Function
- Frequency-Domain Response
  class exercise - Roger's eye frequency-domain response

Motor Circuits



- α-motor neurons initiate motion—they're fast
- each will innervate an average of 200 muscle fibers.
- relatively slow γ-motor neuron regulates muscle tone by setting the reference length of the spindle receptor.
- Golgi tendon organ measures the tension in the tendon and inhibits the α-motor neuron if it exceeds safe levels

Negative Feedback



- If (spindle length > reference), the α-motor neuron cause a contraction of the muscle tissue
- if (spindle length < reference), the α-motor neuron is inhibited, allowing the muscle to extend

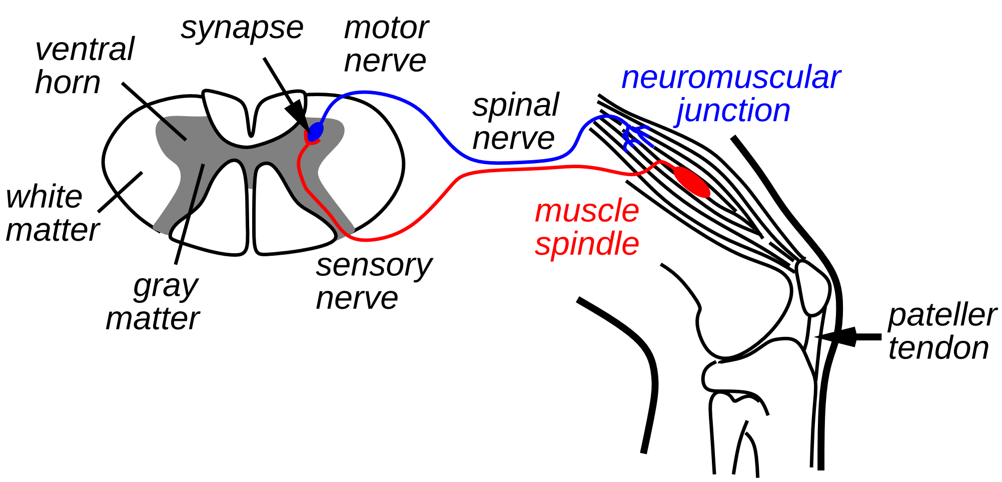
Negative Feedback

...the α-motor neuron changes its output so as to cancel some of its input...

Negative Feedback

- first submitted for a patent in 1928 by Harold S. Black
- it explained the operating principle of many devices including Watt's governor that pre-dated it by some 40 years.
- catalyzed the field of cybernetics
- now heralded as the fundamental principle of stability in compensated dynamical systems

The Muscle Stretch Reflex

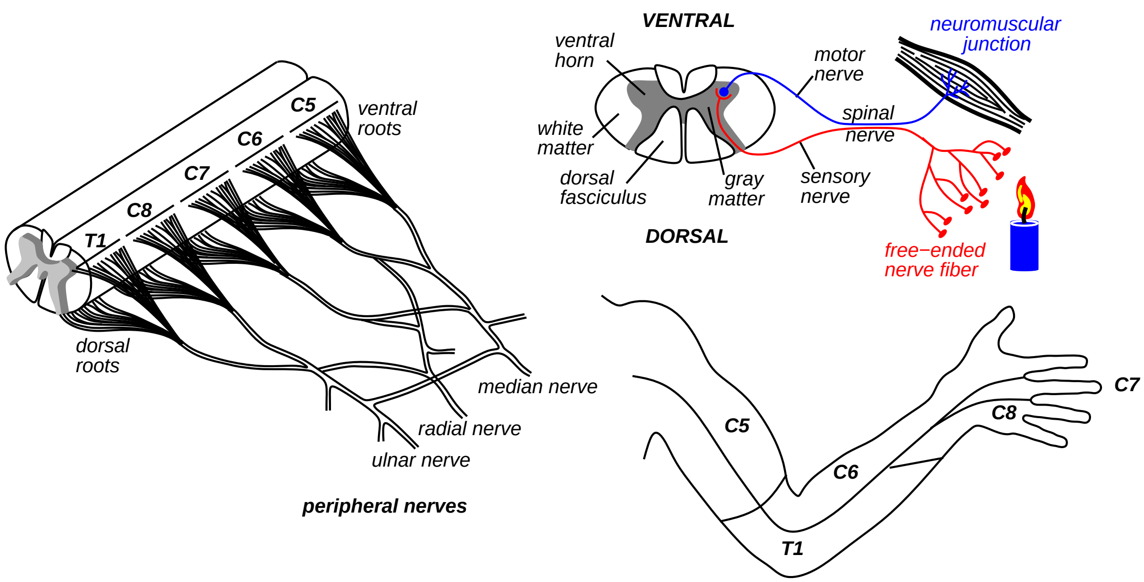


Open- and Closed-Loop Control

open-loop -

a trigger event causes a response without further stimulation

withdrawl reflex



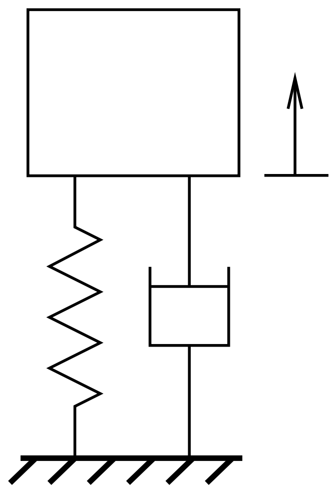
closed-loop -

a (time-varying) setpoint is achieved by constantly measuring and correcting in order to actively reject disturbances

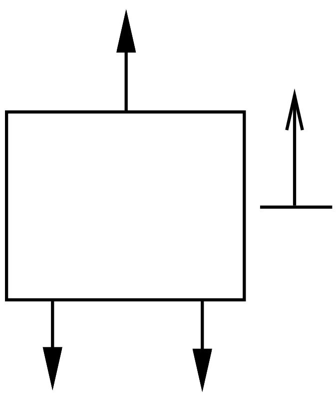
Norbert Weiner - cybernetics (helmsman), homeostasis, endocrine system

The Spring-Mass-Damper

$F_k = -Kx$



$F_b = -Bv = -B\dot{x}$



$$\sum F = m\ddot{x} = f(t) - B\dot{x} - Kx$$

$$m\ddot{x} + B\dot{x} + Kx = f(t), \qquad \text{or}$$

$$\ddot{x} + (B/m)\dot{x} + (K/m)x = f(t)/m = \tilde{f}(t) \quad \text{or}$$

$$\begin{aligned} \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x &= \tilde{f}(t) \qquad \textit{harmonic oscillator} \\ &= 0 \qquad \textit{characteristic equation} \end{aligned}$$

where:

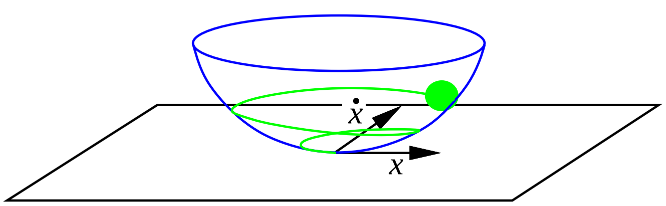
$$\begin{aligned} \omega_n &= (K/m)^{1/2} \qquad [rad/sec] \text{ - } \textit{natural frequency} \\ \zeta &= B/2(Km)^{1/2} \qquad 0 \leq \zeta \leq \infty \text{ - } \textit{damping ratio} \end{aligned}$$

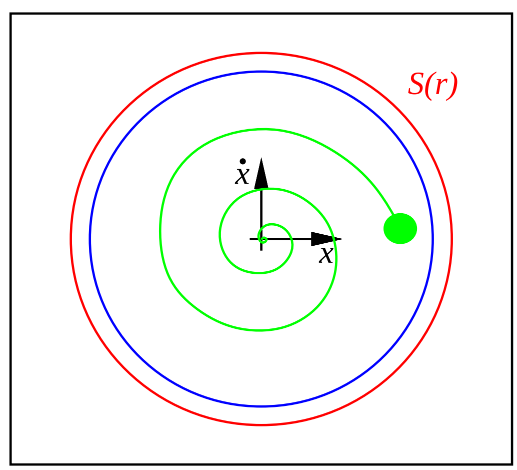
$x'(t) = x(t) - x_{ref}$ *accounts for arbitrary reference positions*

Analytic Stability — Lyapunov's Second/Direct Method

Stability - the origin of the state space is stable if there exists a region, $S(r)$, such that states which start within $S(r)$ remain within $S(r)$.

Asymptotic Stability - a system is asymptotically stable in $S(r)$ if as $t \to \infty$, the system state approaches the origin of the state space.





Analytic Stability - Lyapunov's Second/Direct Method

Define: an arbitrary scalar function, $V(\mathbf{x}, t)$, called a *Lyapunov function*, continuous is all first derivatives, where $\mathbf{x}$ is the state and t is time,

Iff: If the function, $V(\mathbf{x}, t)$, exists such that:

$$\begin{array}{llll} (a)\ V(0,t) & = & 0,\ \ and & \\ (b)\ V(\mathbf{x},t) & > & 0,\ \ for\ x \neq 0 & (positive\ definite),\ \ and \\ (c)\ \partial V/\partial t & < & 0 & (negative\ definite), \end{array}$$

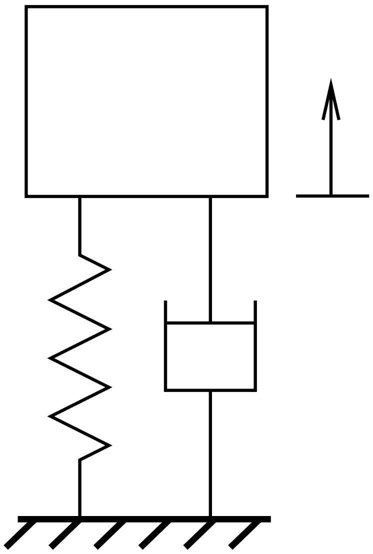
Then: the system described by V is asymptotically stable in the neighborhood of the origin.

...if a system is stable, then there exists a suitable Lyapunov function.

...if, however, a particular Lyapunov function does not satisfy these criteria, it is not necessarily true that this system is unstable.

EXAMPLE: spring-mass-damper

system dynamics:



$$\ddot{x} + \frac{B}{m}\dot{x} + \frac{K}{m}x = 0$$

$$\begin{aligned} E &= \int_0^v (mv)dv + \int_0^x (Kx)dx \\ &= \frac{1}{2}mv^2 + \frac{1}{2}Kx^2 \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2 \end{aligned}$$

Lyapunov function:

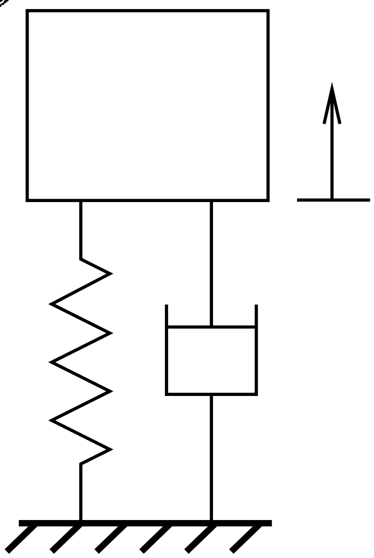
$$V(\mathbf{x}, t) = E = \frac{m\dot{x}^2}{2} + \frac{Kx^2}{2}$$

$(a)\ V(0, t) \ = \ 0, \qquad \surd$

$(b)\ V(\mathbf{x}, t) \ > \ 0, \qquad \surd$

$(c)\ \partial V / \partial t \qquad negative\ definite?$

EXAMPLE: spring-mass-damper



Lyapunov function:

$$V(\mathbf{x}, t) = E = \frac{m\dot{x}^2}{2} + \frac{Kx^2}{2}$$

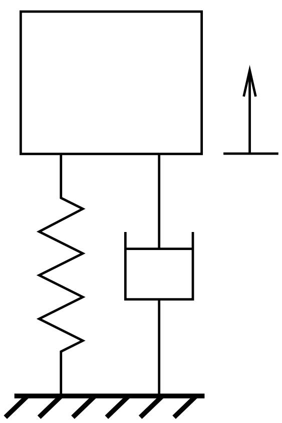
$$\frac{dE}{dt} = m\dot{x}\ddot{x} + Kx\dot{x}$$

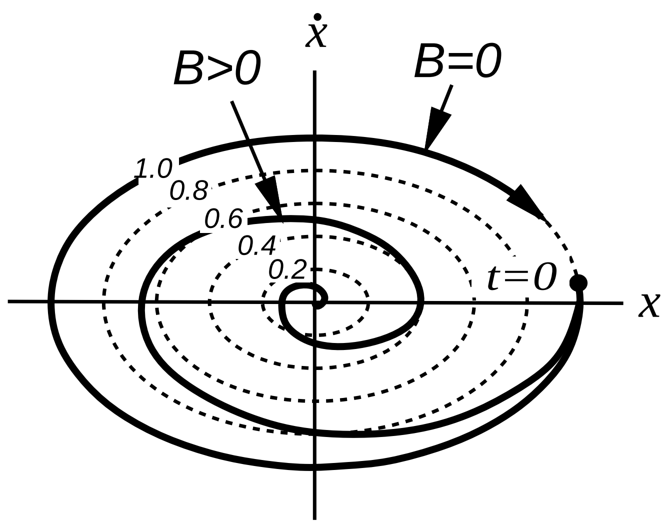
$$\frac{dE}{dt} = m\dot{x}\left[-(B/m)\dot{x} - (K/m)x\right] + Kx\dot{x}$$

$$\frac{dE}{dt} = -B\dot{x}^2$$

stable? or not stable?

EXAMPLE: spring-mass-damper





...the entire state space is asymptotically stable for $B > 0$.

Recap: Introduction to Control

So far, we have:

- introduced the concept of negative feedback in robotics and biology;
- proposed the spring-mass-damper (SMD) as a prototype for proportional-derivative (PD) control;
- we derived the dynamics for the SMD using Newton's laws and a free body diagram; and
- we introduced Lypunov's Direct Method to show the the SMD (and thus PD control) is asymptotically stable.

Now: we describe more tools for analyzing closed-loop linear controllers — the Laplace transform and transfer functions

Tools: Complex Numbers

Cartesian form: $s = \sigma + j\omega$

- $\sigma = Re(s)$ is the *real* part of s
- $\omega = Im(s)$ is the *imaginary* part of s
- $j = \sqrt{-1}$ (sometimes I may use i)

Polar form: $s = re^{j\phi}$

- $r = \sqrt{\sigma^2 + \omega^2}$ is the *modulus* or *magnitude* of s
- $\phi = atan(\omega/\sigma)$ is the *angle* or *phase* of s
- Euler's formula: $e^{j\phi} = cos(\phi) + jsin(\phi)$

complex exponential of $s = \sigma + j\omega$:

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t} \left[cos(\omega t) + jsin(\omega t) \right]$$

Laplace Transform

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt \qquad \text{where } s = \sigma + j\omega$$

- $F(s)$ is a complex-valued function of complex numbers
- s is called the complex *frequency* variable in units of $[\frac{1}{sec}]$; t is *time* in $[sec]$; st is unitless

The Laplace integral will converge if:

- $f(t)$ is piecewise continuous,
- $f(t)$ is of exponential order — i.e., there exists an a such that $|f(t)| \leq Me^{at}$ for all $t > T$ where T is some finite time.

Example: Laplace transform of $f(t) = e^t$

$$F(s) = \int_0^\infty e^t e^{-st} dt = \int_0^\infty e^{(1-s)t} dt = \frac{1}{1-s} e^{(1-s)t} \Big|_0^\infty$$

if we assume that $Re(s) > 1$ so that $e^{(1-s)t} \to 0$ as $t \to \infty$, then

$$F(s) = \frac{1}{1-s} \left[e^{(1-s)\infty} - e^{(1-s)0} \right] = \frac{1}{s-1}$$

therefore,

$$\mathcal{L}[e^t] = \frac{1}{s-1}$$

Example: Laplace transform of "unit step"

unit step: $u(t) = 1$ (for $t \geq 0$)

$$F(s) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

therefore,

$$\mathcal{L}[u(t)] = \frac{1}{s}$$

...fortunately, a lot of these examples have already been worked out by other people and published in tables...

Laplace Transform Pairs

Name	$f(t)$	$F(s)$
unit impulse	$\delta(t)$	1
unit step	$u(t)$	$\frac{1}{s}$
ramp	t	$\frac{1}{s^2}$
n^{th}-order ramp	t^n	$\frac{n!}{s^{n+1}}$
exponential	e^{-at}	$\frac{1}{s+a}$
ramped exponential	$\frac{1}{(n-1)!} t^{n-1} e^{-at}$	$\frac{1}{(s+a)^n}$
sine	$\sin at$	$\frac{a}{s^2+a^2}$
cosine	$\cos at$	$\frac{s}{s^2+a^2}$
damped sine	$e^{-at} sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
damped cosine	$e^{-at} sin\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
hyperbolic sine	$\sinh at$	$\frac{a}{s^2-a^2}$
hyperbolic cosine	$\cosh at$	$\frac{s}{s^2-a^2}$

Laplace Transform

...so what does this do for us?

if we assume that the robot movements are functions of time $f(t)$, such that

$$f(t) \sim e^{st}$$

then, from calculus:

$$\frac{d}{dt}\left[f(t)\right] = \dot{f}(t) \;\sim\; se^{st}$$

$$\int f(t)dt \;\sim\; \frac{1}{s}e^{st}$$

let's say this a different way (ignoring some details about boundary conditions for now), if $\mathcal{L}\left[f(t)\right] = F(s)$, then

$$\mathcal{L}\left[\frac{df}{dt}\right] \;=\; sF(s) \text{ , and}$$

$$\mathcal{L}\left[\int f(t)dt\right] \;=\; \frac{1}{s}F(s)$$

Laplace Transform
Differential Equations

for example, $$\dot{f} + af = 0$$

i.e. the "slope" of function f (df/dt) is proportional to the value of the function, $df/dt = -af$

assuming $f(t) \sim e^{st}$:

$$\begin{aligned} sF(s) + aF(s) &= 0 \\ (s+a)F(s) &= 0 \end{aligned}$$

and the first-order differential equation is transformed into polynomial $(s + a)$,

root $(s = -a)$ tells us more about function $f(t)$,

$$f(t) \sim Ae^{-at}$$

where A is a constant that depends on *boundary conditions*, we will look at that in subsequent examples.

Implications for the Harmonic Oscillator

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2\theta = \widetilde{f}_d(t) \quad \underset{\mathcal{L}^{-1}(\cdot)}{\overset{\mathcal{L}(\cdot)}{\rightleftarrows}} \quad \left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] X(s) = \widetilde{F}_d(s)$$

the homogeneous (unforced) form (i.e. when $\widetilde{f}_d = 0$)

yields the *characteristic equation* of the 2^{nd}-order oscillator

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Roots of the Characteristic Equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

roots $\Rightarrow$ values of s in Ae^{st} that satisfy the original differential equation

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$$

$$\begin{aligned} s_{1,2} &= \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{2\omega_n[-\zeta \pm \sqrt{\zeta^2 - 1}]}{2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}, \end{aligned}$$

three cases:

- repeated real roots ($\zeta = 1$)
- distinct real roots ($\zeta > 1$)
- complex conjugates roots ($\zeta < 1$)

Roots of the Characteristic Equation

For two distinct roots

$$x(t) = A_0 + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

the solution in $t \in [0, \infty)$ requires three boundary conditions to solve for three unknowns A_0, A_1, and A_2

$$\begin{aligned} x(0) = x_0 &= A_0 + A_1 + A_2 \\ \dot{x}(0) = \dot{x}_0 &= s_1 A_1 + s_2 A_2, \\ x(\infty) = x_\infty &= A_0 \end{aligned}$$

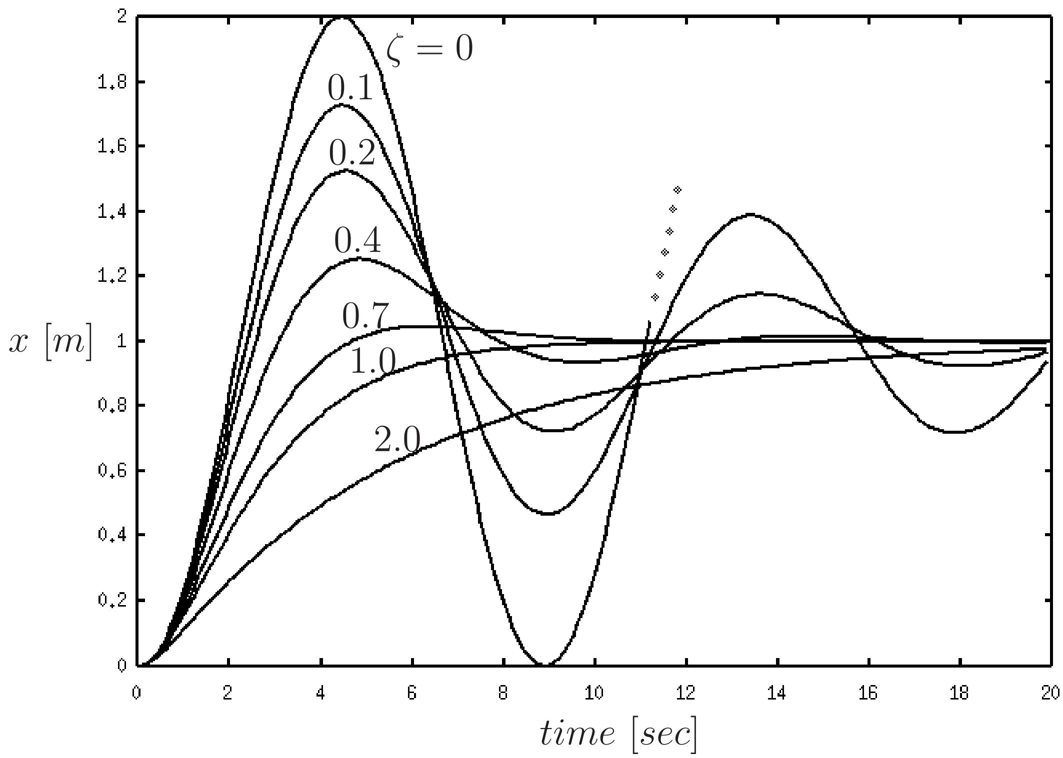
so, a complete time-domain solution is determined

$$x(t) = x_\infty + \frac{(x_0 - x_\infty)s_2 - \dot{x}_0}{s_2 - s_1}\, e^{s_1 t} + \frac{(x_0 - x_\infty)s_1 - \dot{x}_0}{s_1 - s_2}\, e^{s_2 t}$$

Roots of the Characteristic Equation

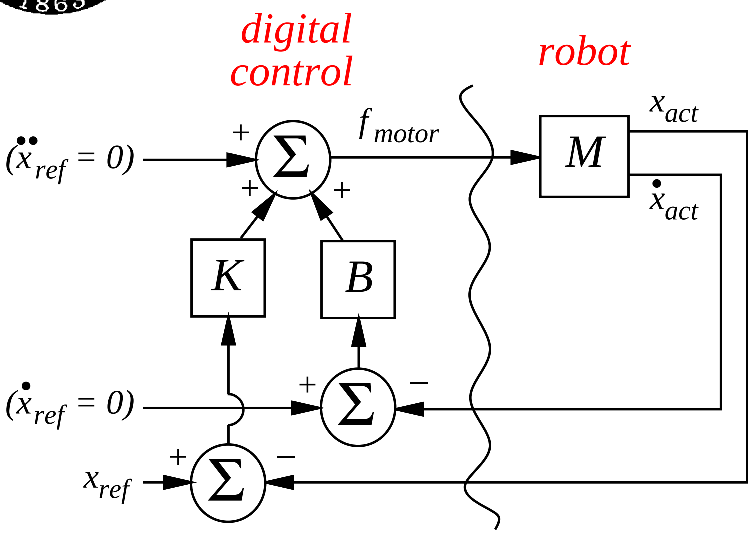
given boundary conditions $x_0 = \dot{x}_0 = 0$ and $x_\infty = 1.0$ the solution simplifies to

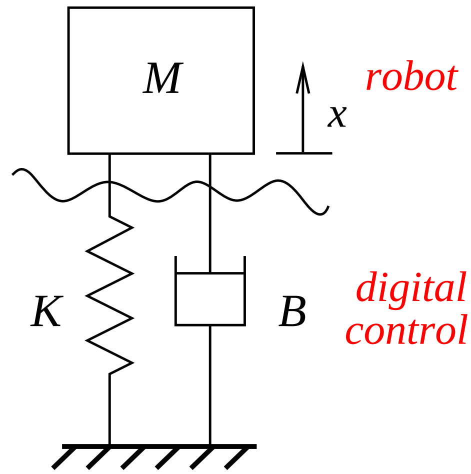
$$x(t) = 1.0 - \frac{s_2}{s_2 - s_1} \; e^{s_1 t} - \frac{s_1}{s_1 - s_2} \; e^{s_2 t}$$



$(K = 1.0 \; [N/m], \; M = 2.0 \; [kg])$

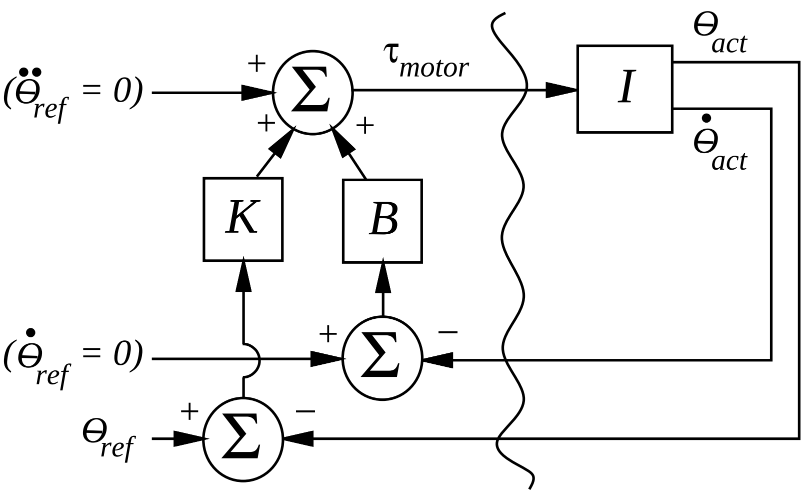
Closed-Loop Control

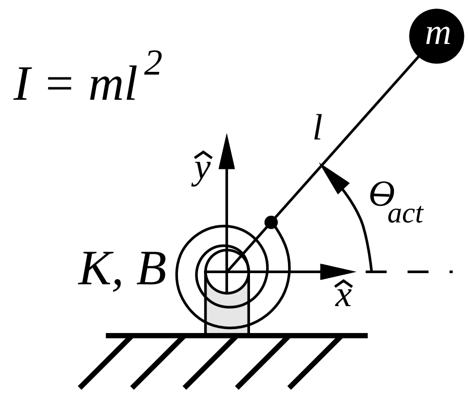




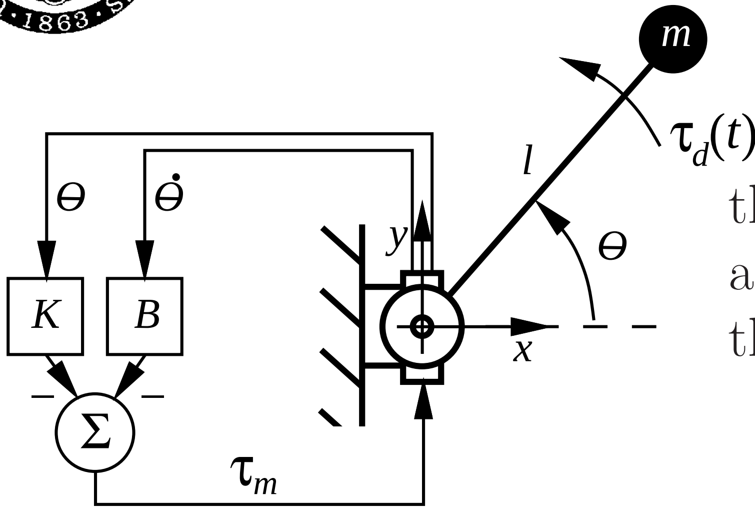
sample and hold $\Delta t = \tau$
where $\frac{1}{\tau}$ [Hz] is the servo rate

analog $\Delta t \rightarrow 0$





A Robot Controller



the controller samples θ and $\dot{\theta}$ and drives the motor to emulate the analog spring and damper

$$\tau_m = -B\dot{\theta} - K\theta$$

$$\sum \tau = I\ddot{\theta} = \tau_d + \tau_m = \tau_d - B\dot{\theta} - K\theta, \quad \text{so that}$$

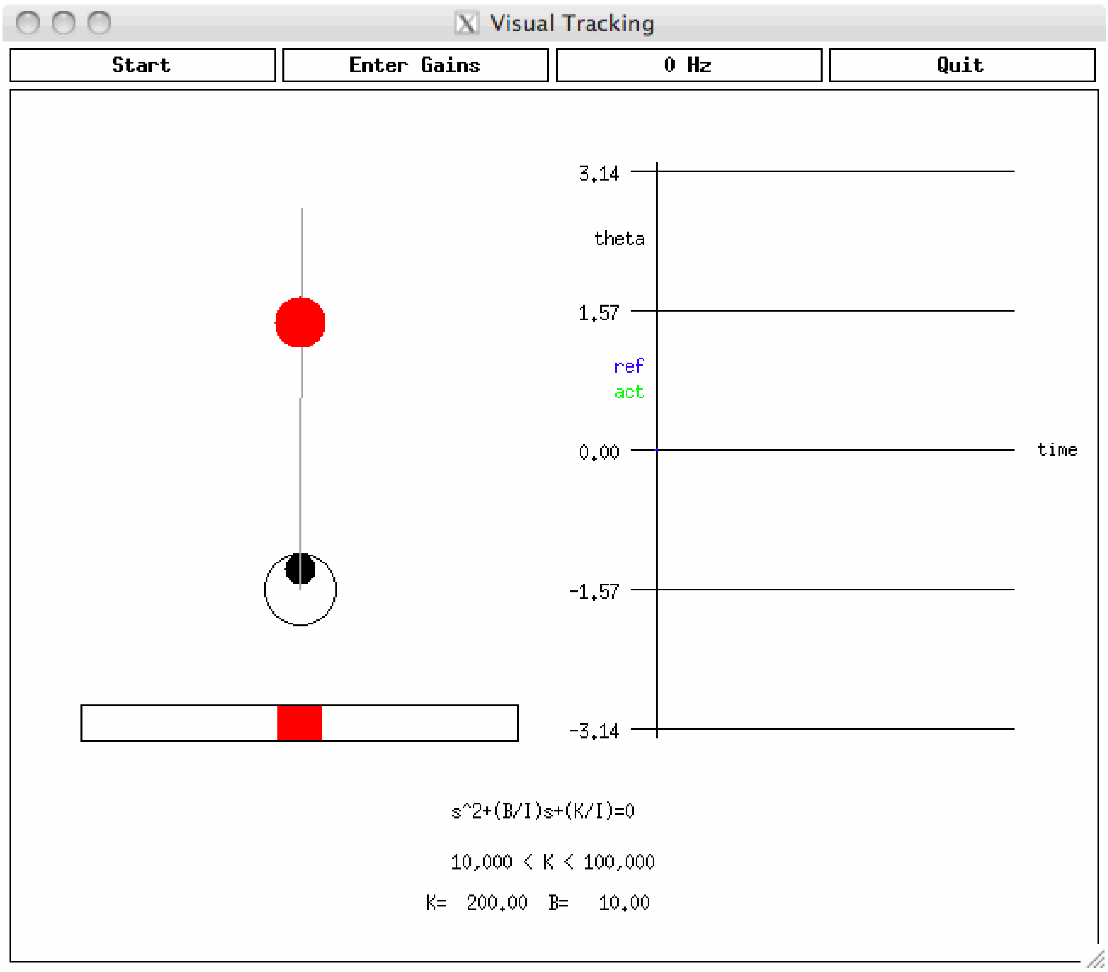
$$I\ddot{\theta} + B\dot{\theta} + K\theta = \tau_d$$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \tilde{\tau}_d$$

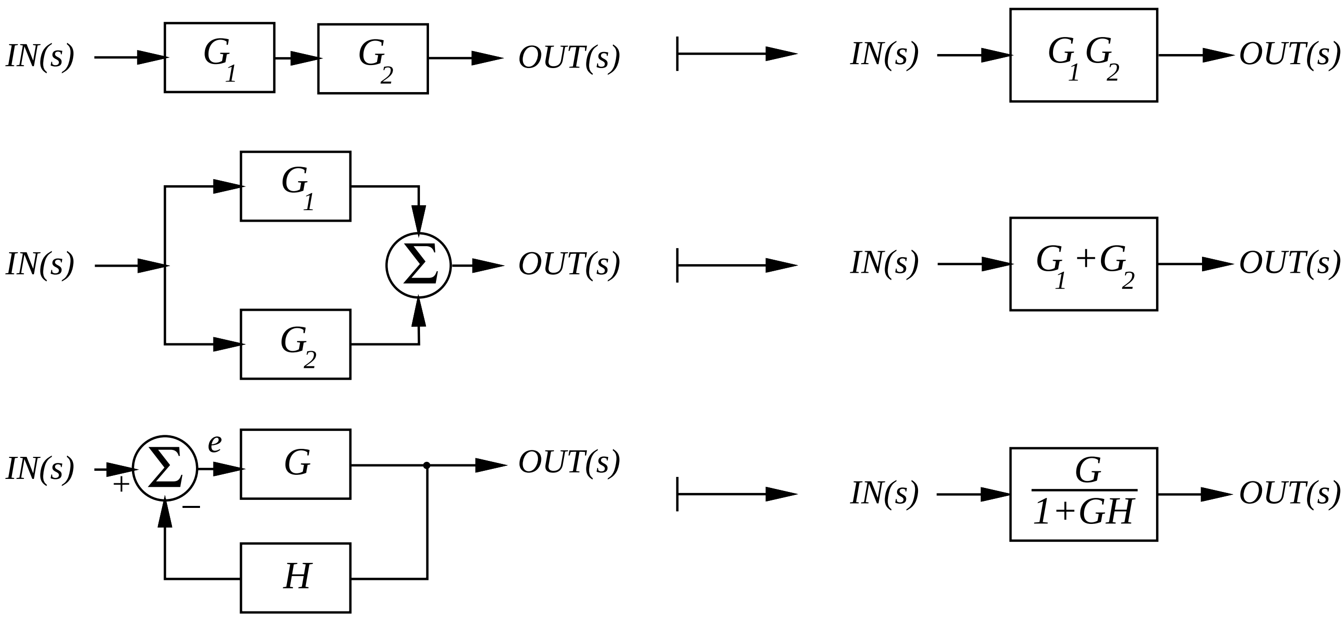
where, in this case,

$$\zeta = \frac{B}{2\sqrt{KI}}, \quad \text{and} \quad \omega_n = \sqrt{K/I}$$

Class Exercise



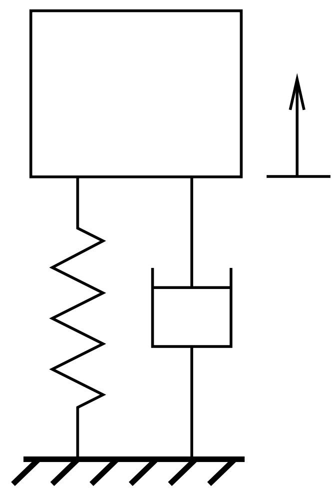
Transfer Functions



$$
\begin{aligned}
IN(s) - OUT(s)H(s) &= e(s) = \frac{OUT(s)}{G(s)} \\
IN(s) &= OUT(s)\left[\frac{1}{G(s)} + H(s)\right] = OUT(s)\left[\frac{1+G(s)H(s)}{G(s)}\right] \\
\frac{OUT(s)}{IN(s)} &= \frac{G(s)}{1+G(s)H(s)}
\end{aligned}
$$

closed-loop transfer function

Spring-Mass-Damper
Closed-Loop Transfer Function



$$\tilde{f}(t) = \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x$$
$$\tilde{F}(s) = \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right) X(s),$$

so that, we can write it in the form of a
closed-loop transfer function

$$\frac{X(s)}{\tilde{F}(s)} = \boxed{\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

$$\tilde{F}_{in}(s) \rightarrow \boxed{\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}} \rightarrow X_{out}(s)$$

...with a change of variable, we can re-write this transfer function to accept a position reference input...

Spring-Mass-Damper Equilibrium Setpoint Control

...note that if we apply a constant force $\tilde{F}(s)$ to the mass, the system will settle into a steady state deflection $X_{ref}(s)$...

$$\tilde{F}(s) = constant = K X_{ref}(s)$$

therefore,

$$K X_{ref}(s) = \left(M s^2 + B s + K \right) X_{act}(s), \quad \text{and,}$$

$$\frac{X_{act}(s)}{X_{ref}(s)} = \frac{K}{M s^2 + B s + K} = \boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

$$X_{ref}(s) \rightarrow \boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \rightarrow X_{act}(s)$$

Solving with the Laplace Transform Tables

The Time Domain Response

...at $t = 0$, apply a unit step reference input

$$x_{ref}(t) = 1$$

$$X_{ref}(s) = \frac{1}{s}$$

Therefore, if we let $\omega_n = 1$ and $\zeta = 1$

$$X_{act}(s) = \left[\frac{1}{s^2 + 2s + 1}\right]\left[\frac{1}{s}\right] = \frac{1}{s(s+1)^2}$$

partial-fraction expansion of this quotient yields:

$$\begin{aligned} X_{act}(s) &= \frac{1}{s(s+1)^2} = \frac{a}{s} + \frac{b}{(s+1)} + \frac{c}{(s+1)^2} \\ &= \frac{1}{s} + \frac{-1}{(s+1)} + \frac{-1}{(s+1)^2} \end{aligned}$$

The inverse Laplace transform (from the tables)

$$x_{act}(t) = 1 - e^{-t} - te^{-t}$$

so that at $t = 0$, $x_{act}(t) = 0$, but as $t \rightarrow \infty$, the robot converges to the reference position.

Frequency-Domain Response

to get some insight into how different input *frequencies* influence the output response, consider a sinusoidal input with frequency ω.

$$x_{ref}(t) = A\cos\omega t \qquad R(s) = \frac{As}{s^2+\omega^2} = \frac{As}{(s-i\omega)(s+i\omega)}$$

and the partial fraction expansion incorporates two more terms

$$C(s) = C_{cltf}(s) + \frac{k_1}{s-i\omega} + \frac{k_2}{s+i\omega}.$$

whose roots $s = \pm i\omega$ are purely imaginary and the inverse Laplace transform of these terms yields time domain responses like:

$$k_1 e^{i\omega t} \text{ and, } k_2 e^{-i\omega t}$$

...the steady state response of the second order system in response to a sinusoidal input is also a contact amplitude sinusoid of the same frequency...

Frequency-Domain Response - continued

the magnitude of the sinusoidal response will be proportional to the amplitude of the forcing function, A, and the gain expressed in the closed-loop transfer function,

$$\frac{G(s)}{1+G(s)H(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{(s/\omega_n)^2+2\zeta(s/\omega_n)+1}$$

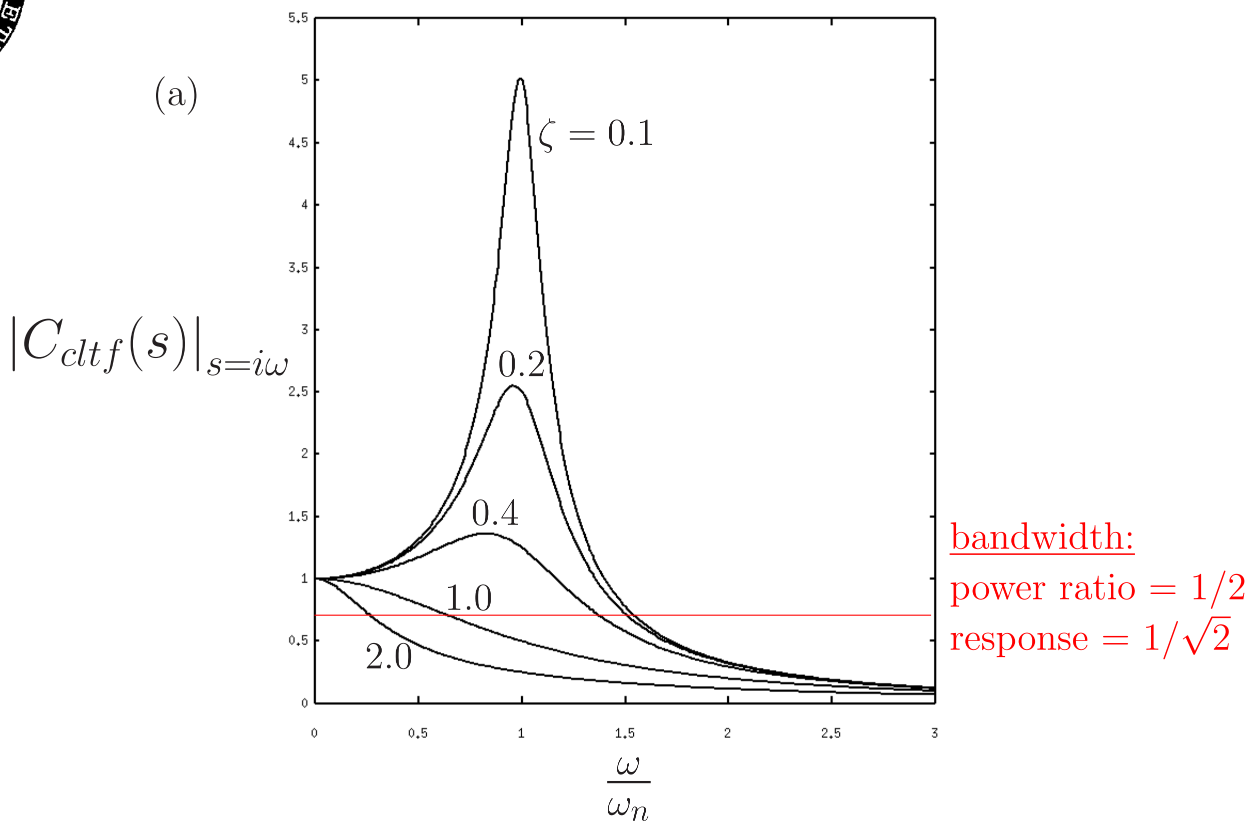
The gain from the CLTF can be determined by evaluating the CLTF at the roots introduced by the forcing function ($s = \pm i\omega$). The result is a complex number with corresponding magnitude and phase:

$$\left|\frac{G(s)}{1+G(s)H(s)}\right|_{s=i\omega} = \frac{1}{\left[(1-(\omega/\omega_n)^2)^2+(2\zeta(\omega/\omega_n))^2\right]^{1/2}}$$

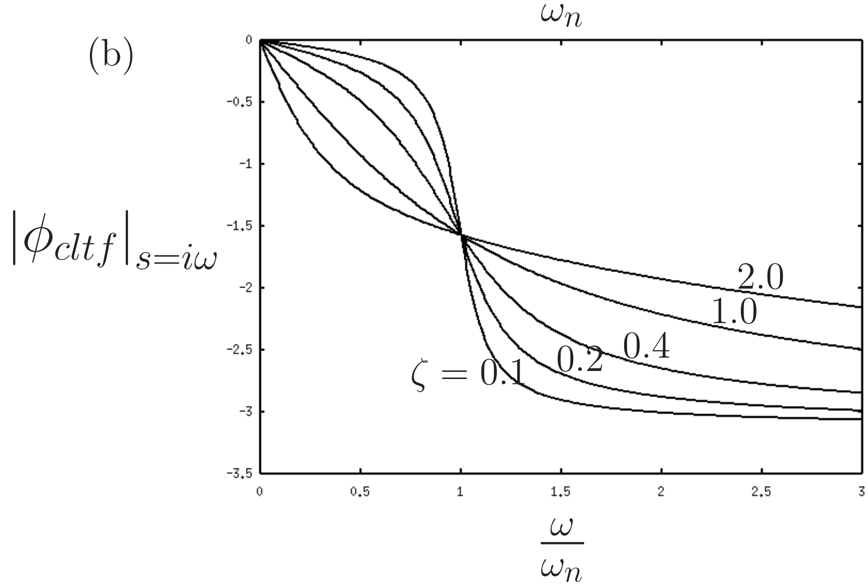
$$\phi(\omega) = -tan^{-1}\left(\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right)$$

Frequency-Domain Response

(a)



(b)



Class Exercise