Basic Tools of Linear Control Theory

Outline

• Spinal Motor Units
• Negative Feedback
• Open- and Closed-Loop Control
• The Spring-Mass-Damper
• Lyapunov Stability
• Laplace Transform
• the Characteristic Equation
• Equilibrium Setpoint Control - A Robot Controller
class exercise - Roger’s eye and PD control
• Frequency-Domain Response Demonstration
Roger’s eye frequency-domain response
• **α-motor neurons** initiate motion—they’re fast

• each will innervate an average of 200 muscle fibers.

• relatively slow **γ-motor neuron** regulates muscle tone by setting the reference length of the spindle receptor.

• Golgi tendon organ measures the tension in the tendon and inhibits the **α-motor neuron** if it exceeds safe levels
If (spindle length > reference), the α-motor neuron cause a contraction of the muscle tissue

if (spindle length < reference), the α-motor neuron is inhibited, allowing the muscle to extend

...the α-motor neuron changes its output so as to cancel some of its input...
Negative Feedback

- first submitted for a patent in 1928 by Harold S. Black
- it explained the operating principle of many devices including Watt’s governor that pre-dated it by some 40 years.
- catalyzed the field of cybernetics
- now heralded as the fundamental principle of stability in compensated dynamical systems

The Muscle Stretch Reflex
Open- and Closed-Loop Control

**open-loop** -
a trigger event causes a response without further stimulation

*withdrawl reflex*

**closed-loop** -
a (time-varying) setpoint is achieved by constantly measuring and correcting in order to actively reject disturbances

Norbert Weiner - cybernetics (helmsman), homeostasis, endocrine system
The Spring-Mass-Damper

\[ F_k = -Kx \quad \text{or} \quad \ddot{x} + \left(\frac{B}{m}\right) \dot{x} + \left(\frac{K}{m}\right)x = f(t)/m = \ddot{f}(t) \quad \text{“specific” applied force} \]

\[ \ddot{x} + (B/m)\dot{x} + (K/m)x = 0 \quad \text{the characteristic equation} \]

arbitrary references require a change of variables:

\[ x'(t) = x(t) - x_{ref} \]
\[ \dot{x}' = \dot{x} \]
The Spring-Mass-Damper

\[ m \]

\[ K \]

\[ B \]

\[ x = 0 \]

\[ \ddot{x} + \left( \frac{B}{m} \right) \dot{x} + \left( \frac{K}{m} \right) x = 0 \]

we can write this another way:

\[ \ddot{x} + 2 \zeta \omega_n \dot{x} + \omega_n^2 x = 0 \quad \text{harmonic oscillator} \]

where:

\[ \omega_n = \sqrt{\frac{K}{m}} \quad [\text{rad/sec}] - \text{natural frequency} \]

\[ \zeta = \frac{B}{2 \sqrt{Km}} \quad 0 \leq \zeta \leq \infty - \text{damping ratio} \]
Closed-Loop Control

\[ (x_{\text{ref}} = 0) \quad + \quad \sum \quad f_{\text{motor}} \quad \rightarrow \quad M \quad \rightarrow \quad x_{\text{act}} \]

\[ (x_{\text{ref}} = 0) \quad + \quad \sum \quad - \quad x_{\text{ref}} \]

Sample and hold \( \Delta t = \tau \)
Where \( \frac{1}{\tau} [\text{Hz}] \) is the servo rate

\[ \Sigma \quad K \quad B \quad \sum \quad \tau_{\text{motor}} \quad \rightarrow \quad I \quad \rightarrow \quad \dot{\Theta}_{\text{act}} \]

\[ \dot{\Theta}_{\text{ref}} = 0 \quad + \quad \sum \quad \rightarrow \quad I \quad \rightarrow \quad \dot{\Theta}_{\text{act}} \]

\[ I = ml^2 \]

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Roger MotorUnits.c
Master Control Procedure

/* == the simulator executes control_roger() once ==*/
/* == every simulated 0.001 second (1000 Hz) ==*/
control_roger(roger, time)
Robot * roger;
double time;
{
    update_setpoints(roger);

    // turn setpoint references into torques
    PDController_base(roger, time);
    PDController_arms(roger, time);
    PDController_eyes(roger, time);
}
double Kp_eye, Kd_eye;
// gain values set in enter_params()

/* Eyes PD controller: */
/* -pi/2 < eyes_setpoint < pi/2 for each eye */
PDController_eyes(roger, time)
Robot * roger;
double time;
{
    int i;
    double theta_error;

    for (i = 0; i < NEYES; i++) {
        theta_error = roger->eyes_setpoint[i] - roger->eye_theta[i];
        // roger->eye_torque[i] = ...
    }
}
double Kp_arm, Kd_arm;
// gain values set in enter_params()

/*  Arms PD controller: -pi < arm_setpoint < pi */
/* for the shoulder and elbow of each arm */
PDController_arms(roger, time)
Robot * roger;
double time;
{
    int i;
    double theta_error;

    for (i = LEFT; i <= RIGHT; ++i) {
        theta_error = roger->arm_setpoint[i][0]
                     - roger->arm_theta[i][0];

        // -M_PI < theta_error < +M_PI

        // roger->arm_torque[i][0] = ...
        // roger->arm_torque[i][1] = ...
    }
}
Analytic Stability —
Lyapunov’s Second/Direct Method

**Stability** - the origin of the state space is stable if there exists a region, $S(r)$, such that states which start within $S(r)$ remain within $S(r)$.

**Asymptotic Stability** - a system is asymptotically stable in $S(r)$ if as $t \to \infty$, the system state approaches the origin of the state space.
Define: an arbitrary scalar function, $V(x, t)$, called a Lyapunov function, continuous is all first derivatives, where $x$ is the state and $t$ is time,

Iff: If the function, $V(x, t)$, exists such that:

(a) $V(0, t) = 0$, and
(b) $V(x, t) > 0$, for $x \neq 0$ (positive definite), and
(c) $\partial V/\partial t < 0$ (negative definite),

Then: the system described by $V$ is asymptotically stable in the neighborhood of the origin.

...if a system is stable, then there exists a suitable Lyapunov function.

...if, however, a particular Lyapunov function does not satisfy these criteria, it is not necessarily true that this system is unstable.
EXAMPLE: spring-mass-damper

system dynamics:

\[ \ddot{x} + \frac{B}{m} \dot{x} + \frac{K}{m} x = 0 \]

\[ x = 0 \]

\[ E = \int_0^v (mv) dv + \int_0^x (Kx) dx \]

\[ = \frac{1}{2}mv^2 + \frac{1}{2}Kx^2 \]

\[ = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2 \]

Lyapunov function:

\[ V(x, t) = E = \frac{m\dot{x}^2}{2} + \frac{Kx^2}{2} \]

(a) \( V(0, t) = 0 \), \( \checkmark \)

(b) \( V(x, t) > 0 \), \( \checkmark \)

(c) \( \partial V/\partial t \) \( \text{negative definite?} \)
EXAMPLE: spring-mass-damper

\[ V(x, t) = E = \frac{m\dot{x}^2}{2} + \frac{Kx^2}{2} \]

\[
\frac{dE}{dt} = m\ddot{x} + Kx\dot{x}
\]

\[
\frac{dE}{dt} = m\dot{x} \left[ -(B/m)\dot{x} - (K/m)x \right] + Kx\dot{x}
\]

\[
\frac{dE}{dt} = -B\dot{x}^2
\]

stable? or not stable?
EXAMPLE: spring-mass-damper

\[ K B x = 0 \]

...the entire state space is asymptotically stable for \( B > 0 \).
Recap: Introduction to Control

So far, we have:

- introduced the concept of negative feedback in robotics and biology;
- proposed the spring-mass-damper (SMD) as a prototype for proportional-derivative (PD) control;
- we derived the dynamics for the SMD using Newton’s laws and a free body diagram; and
- we introduced Lyapunov’s Direct Method to show the SMD (and thus PD control) is asymptotically stable.

Now: we describe more tools for analyzing closed-loop linear controllers — the Laplace transform and transfer functions
Tools: Complex Numbers

Cartesian form:  \( s = \sigma + j\omega \)

- \( \sigma = Re(s) \) is the real part of \( s \)
- \( \omega = Im(s) \) is the imaginary part of \( s \)
- \( j = \sqrt{-1} \) (sometimes I may use \( i \))

Polar form:  \( s = re^{j\phi} \)

- \( r = \sqrt{\sigma^2 + \omega^2} \) is the modulus or magnitude of \( s \)
- \( \phi = \text{atan}(\omega/\sigma) \) is the angle or phase of \( s \)
- Euler’s formula:  \( e^{j\phi} = \cos(\phi) + j\sin(\phi) \)

Therefore, complex exponential of \( s = \sigma + j\omega \):

\[
e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t} [\cos(\omega t) + j\sin(\omega t)]
\]
Laplace Transform

...so what does this do for us?

if we assume that the robot movements are functions of time $f(t)$, such that

$$f(t) \sim e^{st}$$

then, from calculus:

$$\frac{d}{dt}[f(t)] = f'(t) \sim se^{st}$$

$$\int f(t)dt \sim \frac{1}{s}e^{st}$$

let’s say this a different way (ignoring some details about boundary conditions for now), if $\mathcal{L}[f(t)] = F(s)$, then

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) , \text{ and}$$

$$\mathcal{L}\left[\int f(t)dt\right] = \frac{1}{s}F(s)$$
Laplace Transform
Differential Equations

for example,
\[
\dot{f} + af = 0
\]
i.e. the “slope” of function \( f \) \((df/dt)\) is proportional to the value of the function, \( df/dt = -af \)

assuming \( f(t) \sim e^{st} \):

\[
\begin{align*}
    sF(s) + aF(s) &= 0 \\
    (s + a)F(s) &= 0
\end{align*}
\]

and the first-order differential equation is transformed into polynomial \((s + a)\),

root \((s = -a)\) tells us more about function \( f(t) \),

\[
f(t) \sim A_0 + A_1 e^{-at}
\]

where coefficients \( A_0 \) and \( A_1 \) are constants that depend on boundary conditions, i.e. if \( f(0) = 1 \) and \( f(\infty) = 0 \)

verify that: \( A_0 = 0 \) and \( A_1 = 1 \)
The Harmonic Oscillator

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 0 \]
\[ \xrightarrow{\mathcal{L}(\cdot)} \left[ s^2 + 2\zeta \omega_n s + \omega_n^2 \right] X(s) = 0 \]

yields the \textit{characteristic equation} of the 2\textsuperscript{nd}-order oscillator in the complex frequency domain

\[ s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \]
Roots of the Characteristic Equation

\[ s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \]

roots ⇒ values of \( s \) in \( Ae^{st} \) that satisfy the original differential equation

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 0 \]

\[
\begin{align*}
    s_{1,2} &= \frac{-2\zeta \omega_n \pm \sqrt{(2\zeta \omega_n)^2 - 4\omega_n^2}}{2} \\
    &= \frac{2\omega_n [\mp \zeta \pm \sqrt{\zeta^2 - 1}]}{2} \\
    &= -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1},
\end{align*}
\]

three cases:

- repeated real roots (\( \zeta = 1 \))
- distinct real roots (\( \zeta > 1 \))
- complex conjugates roots (\( \zeta < 1 \))
The Root Locus Diagram

\[ K = 1.0 \ [Nm/rad] \]
\[ I = 2.0 \ [kgm^2], \text{ so that} \]
\[-3.5 \leq B \leq 3.5, \text{ or} \]
\[-1.24 \leq \zeta \leq 1.24 \]

\[ \ddot{\theta} + \frac{B}{I} \dot{\theta} + \frac{K}{I} \theta = 0, \quad \text{or} \]
\[ \ddot{\theta} + \frac{B}{2} \dot{\theta} + \frac{1}{2} \theta = 0 \]
Roots of the Characteristic Equation

For two distinct roots

\[ x(t) = A_0 + A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

the solution in \( t \in [0, \infty) \) requires three boundary conditions to solve for three unknowns \( A_0, A_1, \) and \( A_2 \)

\[ 
\begin{align*}
  x(0) &= x_0 = A_0 + A_1 + A_2 \\
  \dot{x}(0) &= \dot{x}_0 = s_1 A_1 + s_2 A_2, \\
  x(\infty) &= x_\infty = A_0
\end{align*} 
\]

so, a complete time-domain solution is determined

\[ x(t) = x_\infty + \frac{(x_0 - x_\infty)s_2 - \dot{x}_0}{s_2 - s_1} e^{s_1 t} + \frac{(x_0 - x_\infty)s_1 - \dot{x}_0}{s_1 - s_2} e^{s_2 t} \]
Roots of the Characteristic Equation

given boundary conditions \( x_0 = \dot{x}_0 = 0 \) and \( x_\infty = 1.0 \) the solution simplifies to

\[
x(t) = 1.0 - \frac{s_2}{s_2 - s_1} e^{s_1 t} - \frac{s_1}{s_1 - s_2} e^{s_2 t}
\]

\[\zeta = 0\]

\[K = 1.0 \; [N/m], \; M = 2.0 \; [kg]\]
Class Exercise

\[ K = 200.00 \quad b = 10.00 \]

\[ \theta^{-2} + \frac{2}{3} + \frac{3}{2} = 0 \]

\[ 10,000 < K < 100,000 \]
Frequency-Domain Response

\[ |C_{cltf}(s)|_{s=i\omega} \]

\[ \text{bandwidth: power ratio } = \frac{1}{2} \]
\[ \text{response: } \frac{1}{\sqrt{2}} \]

\[ \zeta = 0.1 \]
\[ \zeta = 0.2 \]
\[ \zeta = 0.4 \]

\[ \phi_{cltf}, \frac{\omega}{\omega_n} \]

\[ \zeta = 0.1 \]
\[ \zeta = 0.2 \]
\[ \zeta = 0.4 \]
Class Exercise

\[ e^{-\frac{3}{2}} + \frac{\theta}{2} = 0 \]

\( 10,000 < K < 100,000 \)

\( K = 20,000.00 \quad b = 200.00 \)