Dynamics

The branch of physics that treats the action of force on bodies in motion or at rest; kinetics, kinematics, and statics, collectively. — Websters dictionary

Outline

- Conservation of Momentum
- Moment of Inertia, Inertia Tensors
- Newton/Euler Dynamics
- the Computed Torque Equation
- Applications:
  - simulation;
  - control - feedforward compensation;
  - planning;
  - analysis - the acceleration ellipsoid.

Newton's Laws

1. a particle will remain in a state of constant rectilinear motion unless acted on by an external force;

2. the time-rate-of-change in the momentum (mv) of the particle is proportional to the externally applied forces, $F = \frac{d}{dt}(mv)$; and

3. any force imposed on body A by body B is reciprocated by an equal and opposite reaction force on body B by body A.

Conservation of Momentum

Linear:

$$F = \frac{d}{dt}[m\dot{x}] = m\ddot{x}$$

$$[\text{N}] = \left[\frac{\text{kg m}}{\text{sec}^2}\right]$$

Angular:

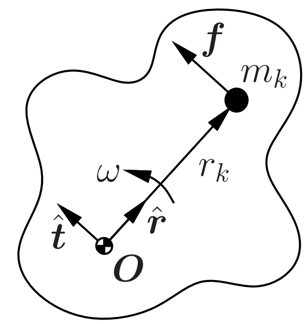
$$\tau = \frac{d}{dt}\left[I\dot{\theta}\right] = I\ddot{\theta}$$

$$[\text{Nm}] = \left[\frac{\text{kg m}^2}{\text{sec}^2}\right]$$

J is called the mass moment of inertia

Conservation of Momentum

To generate an angular acceleration about $\boldsymbol{O}$, a torque is applied around the $\hat{\boldsymbol{z}}$ axis



$$\begin{aligned}\boldsymbol{\tau}_k &= \boldsymbol{r} \times \boldsymbol{f} = r_k \hat{\boldsymbol{r}} \times \frac{d}{dt}(m_k \boldsymbol{v}_k) \\ &= m_k r_k \left[\hat{\boldsymbol{r}} \times \frac{d}{dt}(\boldsymbol{v}_k) \right]\end{aligned}$$

the velocity of m_k due to ω_O is

$$\boldsymbol{v}_k = (\omega \hat{\boldsymbol{z}} \times r_k \hat{\boldsymbol{r}}) = (r_k \omega) \hat{\boldsymbol{t}},$$

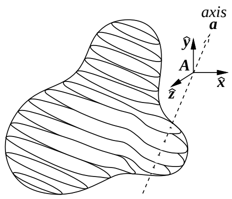
so that

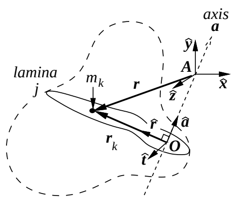
$$\boldsymbol{\tau}_k = (m_k r_k^2) \dot{\omega} \ \hat{\boldsymbol{z}} = J_k \dot{\omega} \ \hat{\boldsymbol{z}}$$

$$\boldsymbol{\tau} = \left(\sum_k m_k r_k^2 \right) \dot{\omega} = J \dot{\omega}.$$

J $[kg \cdot m^2]$ is the scalar *moment of inertia* of the lamina about the $\hat{\boldsymbol{z}}_O$ axis.

Inertia Tensor





$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

MASS MOMENTS OF INERTIA

$$I_{xx} = \int\int\int (y^2 + z^2)\rho dv$$

$$I_{yy} = \int\int\int (x^2 + z^2)\rho dv$$

$$I_{zz} = \int\int\int (x^2 + y^2)\rho dv$$

MASS PRODUCTS OF INERTIA

$$I_{xy} = \int\int\int xy\rho dv$$

$$I_{xz} = \int\int\int xz\rho dv$$

$$I_{yz} = \int\int\int yz\rho dv$$

Inertial Coordinate Frames

Definition (Inertial Frame)

- a frame observation from which Newton's laws apply;
- a frame of reference that is not accelerating (it maintains a constant state of motion) unless subjected to a non-zero force;
- all inertial frames are in a state of constant, rectilinear motion with respect to one another;
- accelerometers moving with any inertial frame detect zero acceleration.

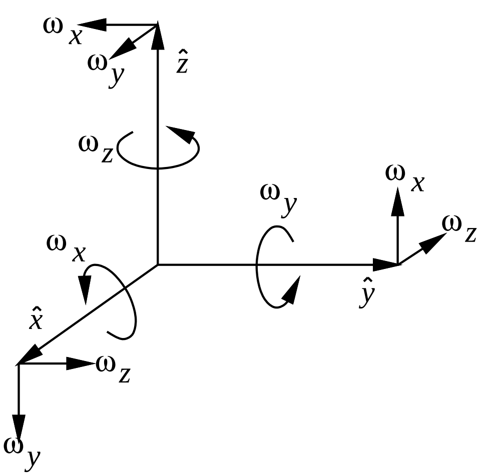
from a non-inertial frame, the dynamics observed depend on the acceleration of that frame and requires that physical (Newtonian) forces must be supplemented by ***fictitious forces*** *to explain the motion.*

Rotating Coordinate Systems

Consider inertial frame A and rotating frame B with absolute velocity $\boldsymbol{\omega}_B$ (written in frame B coordinates).

$$\mathbf{r}_A(t) = {}_AR_B(t)\mathbf{r}_B(t)$$

$$\dot{\mathbf{r}}_A(t) = {}_AR_B(t)\frac{d}{dt}[\mathbf{r}_B(t)] + \frac{d}{dt}[\ {}_AR_B(t)\ ]\mathbf{r}_B(t)$$



To evaluate the second term on the right, consider how the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, basis vectors for frame B change by virtue of $\boldsymbol{\omega}_B$.

$$\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \qquad\qquad \omega_z\hat{\mathbf{y}} \quad -\omega_y\hat{\mathbf{z}} \\
\dot{\hat{\mathbf{y}}} &= -\omega_z\hat{\mathbf{x}} \qquad\qquad -\omega_x\hat{\mathbf{z}} \\
\dot{\hat{\mathbf{z}}} &= \omega_y\hat{\mathbf{x}} \quad -\omega_x\hat{\mathbf{y}}
\end{aligned}$$

this is the cross product written as a matrix operator:

$$\frac{d}{dt}[\ {}_AR_B(t)\ ]\mathbf{r}_B(t) = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$= \boldsymbol{\omega} \times \mathbf{r}$$

Rotating Coordinate Systems

Therefore,

$$\begin{aligned} \dot{\mathbf{r}}_A(t) &= {}_AR_B(t)\frac{d}{dt}[\mathbf{r}_B(t)] + \frac{d}{dt}[\ {}_AR_B(t)\ ]\mathbf{r}_B(t) \\ &= {}_AR_B\left[\dot{\mathbf{r}}_B + (\boldsymbol{\omega}_B^B \times \mathbf{r}_B)\right] \end{aligned}$$

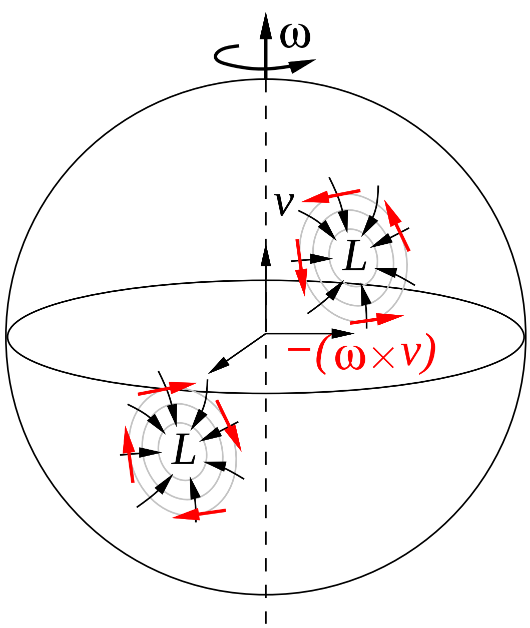
and, in fact, all vector quantities expressed in local frames that are moving relative to an inertial frame are differentiated in this way

$$\frac{d}{dt}\left[{}_A\mathbf{R}_B(t)(\cdot)_B\right] = {}_A\mathbf{R}_B\left[\frac{d}{dt}(\cdot)_B + (\omega_B \times (\cdot)_B)\right]$$

gives rise to the notorious (fictitious)
Coriollis and centrifugal forces!

EXAMPLE:
Rotating Coordinate Systems: Low Pressure Systems

Large scale atmospheric flows converge at low pressure regions. For a nonrotating planet, this would result in flow lines directed radially inward.



but the earth rotates...
consider a stationary inertial frame A and a rotating frame B attached to the earth

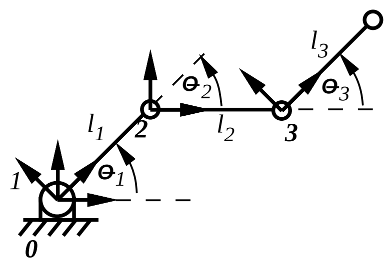
$$\begin{aligned} \boldsymbol{v}_A &= {}_AR_B(t)\boldsymbol{v}_B \\ \dot{\boldsymbol{v}}_A &= {}_AR_B[\dot{\boldsymbol{v}}_B + (\boldsymbol{\omega} \times \boldsymbol{v}_B)] \end{aligned}$$

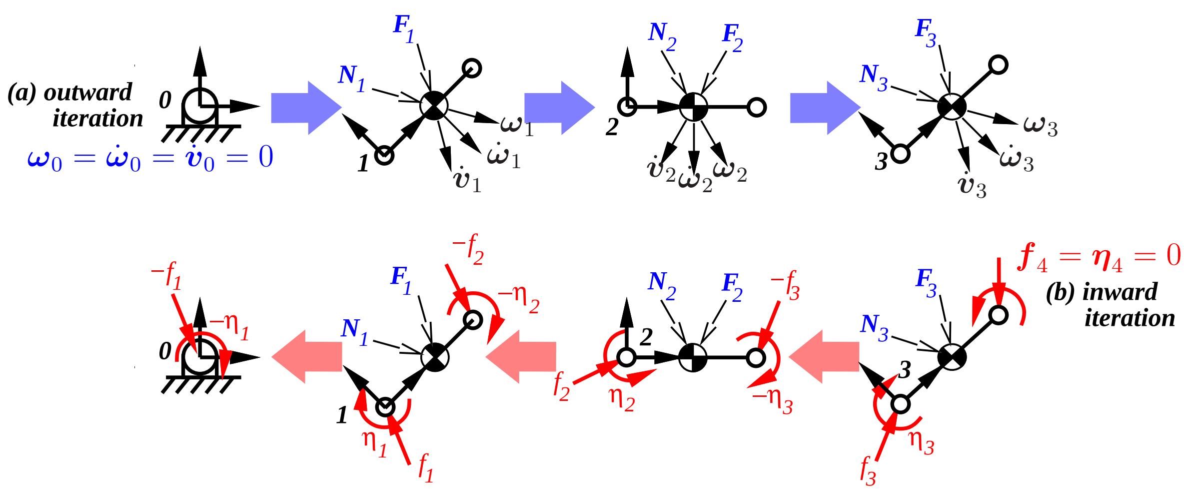
so that to an observer that travels with frame B:

$$\dot{\boldsymbol{v}}_B = {}_BR_A[\dot{\boldsymbol{v}}_A] - (\boldsymbol{\omega} \times \boldsymbol{v}_B)$$

a convergent flow and a rotating system, therefore, leads to a counterclockwise flow in the northern hemisphere and a clockwise rotation in the southern hemisphere.

Newton/Euler Method





the recursive equations for these iterations are derived in Appendix B of the book

Recursive Newton-Euler Equations

Outward Iterations

Angular Velocity: ω

REVOLUTE: $^{i+1}\boldsymbol{\omega}_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\boldsymbol{\omega}_i + \dot{\theta}_{i+1}\hat{\mathbf{z}}_{i+1}$

PRISMATIC: $^{i+1}\boldsymbol{\omega}_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\boldsymbol{\omega}_i$

Angular Acceleration: $\dot{\omega}$

REVOLUTE: $^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\dot{\boldsymbol{\omega}}_i + ({}_{i+1}\mathbf{R}_i \, {}^{i}\boldsymbol{\omega}_i \times \dot{\theta}_{i+1}\hat{\mathbf{z}}_{i+1}) + \ddot{\theta}_{i+1}\hat{\mathbf{z}}_{i+1}$

PRISMATIC: $^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\dot{\boldsymbol{\omega}}_i$

Linear Acceleration: $\dot{v}$

REVOLUTE: $^{i+1}\dot{\boldsymbol{v}}_{i+1} = {}_{i+1}\mathbf{R}_i \left[{}^{i}\dot{\boldsymbol{v}}_i + ({}^{i}\dot{\boldsymbol{\omega}}_i \times {}^{i}\mathbf{p}_{i+1}) + ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{p}_{i+1}) \right]$

PRISMATIC: $^{i+1}\dot{\boldsymbol{v}}_{i+1} = {}_{i+1}\mathbf{R}_i \left[{}^{i}\dot{\boldsymbol{v}}_i + \ddot{d}_i\hat{\mathbf{x}}_i + 2({}^{i}\boldsymbol{\omega}_i \times \dot{d}_i\hat{\mathbf{x}}_i) + ({}^{i}\dot{\boldsymbol{\omega}}_i \times d_i\hat{\mathbf{x}}_i) + ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{\omega}_i \times d_i\hat{\mathbf{x}}_i) \right]$

Linear Acceleration (center of mass): $\dot{v}_{cm}$

$$^{i+1}\dot{\boldsymbol{v}}_{cm,(i+1)} = ({}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} \times {}^{i+1}\mathbf{p}_{cm}) + ({}^{i+1}\boldsymbol{\omega}_{i+1} \times {}^{i+1}\boldsymbol{\omega}_{i+1} \times {}^{i+1}\mathbf{p}_{cm}) + {}^{i+1}\dot{\boldsymbol{v}}_{i+1}$$

Net Force: $\mathbf{F}$

$$^{i+1}\mathbf{F}_{i+1} = m_{i+1} \, {}^{i+1}\dot{\boldsymbol{v}}_{cm \; i+1}$$

Net Moment: $\mathbf{N}$

$$^{i+1}\mathbf{N}_{i+1} = \boldsymbol{I}_{i+1} \, {}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} + ({}^{i+1}\boldsymbol{\omega}_{i+1} \times \boldsymbol{I}_{i+1} \, {}^{i+1}\boldsymbol{\omega}_{i+1})$$

Inward Iterations

Inter-Link Forces:

$$^{i}\mathbf{f}_i = {}^{i}\mathbf{F}_i + {}_{i}\mathbf{R}_{i+1} \, {}^{i+1}\mathbf{f}_{i+1}$$

Inter-Link Moments:

$$^{i}\boldsymbol{\eta}_i = {}^{i}\mathbf{N}_i + {}_{i}\mathbf{R}_{i+1} \, {}^{i+1}\boldsymbol{\eta}_{i+1} + ({}^{i}\mathbf{p}_{cm} \times {}^{i}\mathbf{F}_i) + ({}^{i}\mathbf{p}_{i+1} \times {}_{i}\mathbf{R}_{i+1} \, {}^{i+1}\mathbf{f}_{i+1})$$

The Computed Torque Equation

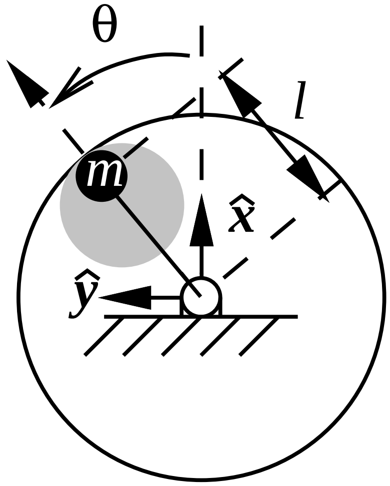
State Space Form

$$\boldsymbol{\tau} = \mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{V}(\boldsymbol{\theta}\ \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta}) + \mathbf{F}$$

external forces/torques:

- external forces
- friction
  - viscous $\tau = -v\dot{\theta}$
  - coulomb $\tau = -c(sgn(\dot{\theta}))$
  - hybrid

EXAMPLE:
Dynamic Model of Roger's Eye



$$\sum \tau = \frac{d}{dt}(J\dot{\theta})$$

$$\tau_m + mglsin(\theta) = (ml^2)\ddot{\theta}$$

or

$$\tau_m = \mathbf{M}\ddot{\theta} + \mathbf{G},$$

generalized inertia

$$\mathbf{M} = ml^2 \text{ (a scalar);}$$

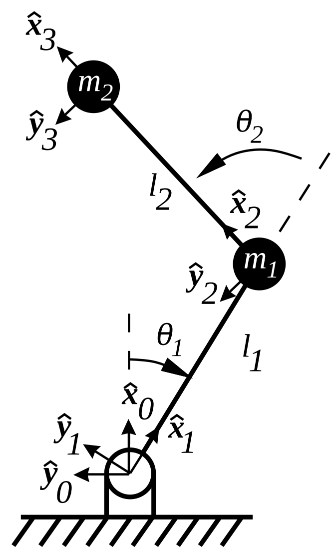
Coriolis and centripetal forces

$$\mathbf{V}(\theta, \dot{\theta}) \text{ do not exist; and}$$

Gravitational loads

$$\mathbf{G} = -mglsin(\theta)$$

EXAMPLE:
Dynamic Model of Roger's Arm

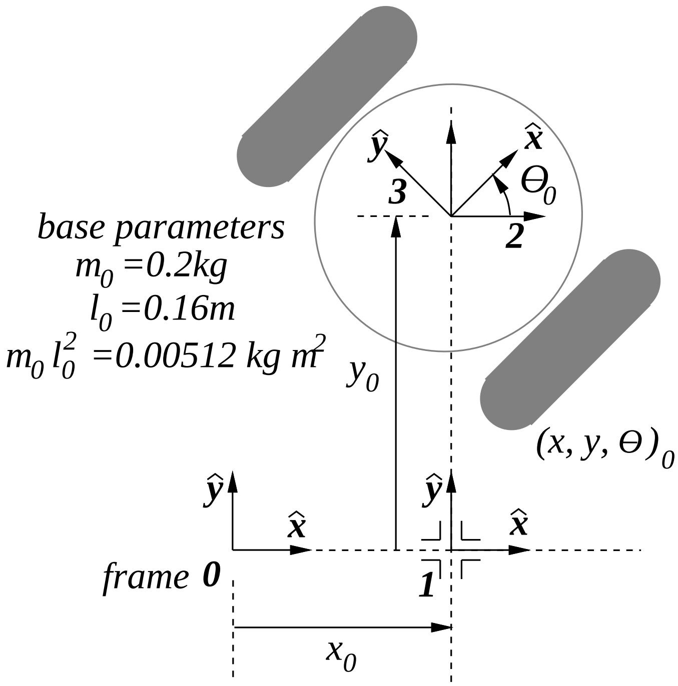


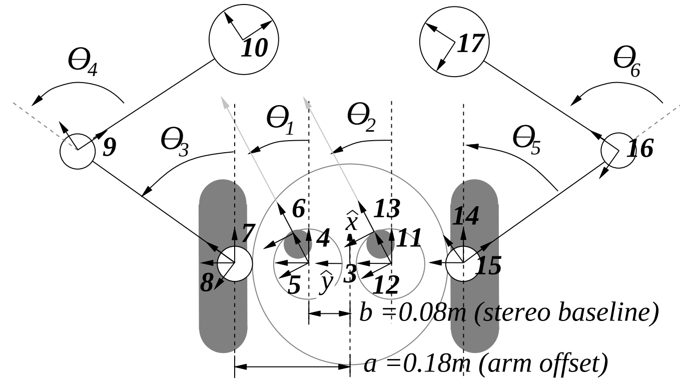
$$\mathbf{M}(\boldsymbol{\theta}) = \begin{bmatrix} m_2 l_2^2 + 2 m_2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix}$$

$$\mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 (\dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2) \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix} \quad Nm$$

$$\mathbf{G}(\boldsymbol{\theta}) = \begin{bmatrix} -(m_1 + m_2) l_1 s_1 g - m_2 l_2 s_{12} g \\ -m_2 l_2 s_{12} g \end{bmatrix} \quad Nm$$

EXAMPLE:
Roger's Whole-Body Dynamics





$$\tilde{\boldsymbol{\tau}} = \mathbf{M}\ddot{\boldsymbol{q}} + \mathbf{V}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{F}$$

$\tilde{\boldsymbol{\tau}} \in \mathbb{R}^8$ is the vector of forces and torques that cause accelerations in the degrees of freedom $\boldsymbol{q} \in \mathbb{R}^8$ of the robot.

$$\begin{bmatrix} \tilde{\tau}_0 \\ \tilde{\tau}_1 \\ \tilde{\tau}_2 \\ \tilde{\tau}_3 \\ \tilde{\tau}_4 \\ \tilde{\tau}_5 \\ \tilde{\tau}_6 \\ \tilde{\tau}_7 \end{bmatrix} = \begin{bmatrix} M_{00} & M_{01} & M_{02} & M_{03} & M_{04} & M_{05} & M_{06} & M_{07} \\ M_{10} & M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} & M_{17} \\ M_{20} & M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} & M_{27} \\ M_{30} & M_{31} & M_{32} & M_{33} & M_{34} & M_{35} & M_{36} & M_{37} \\ M_{40} & M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} & M_{47} \\ M_{50} & M_{51} & M_{52} & M_{53} & M_{54} & M_{55} & M_{56} & M_{57} \\ M_{60} & M_{61} & M_{62} & M_{63} & M_{64} & M_{65} & M_{66} & M_{67} \\ M_{70} & M_{71} & M_{72} & M_{73} & M_{74} & M_{75} & M_{76} & M_{77} \end{bmatrix} \begin{bmatrix} \ddot{q}_0 \\ \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \\ \ddot{q}_5 \\ \ddot{q}_6 \\ \ddot{q}_7 \end{bmatrix} + \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} + \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{3x} \\ \boldsymbol{\eta}_{3z} \\ \boldsymbol{\eta}_{5z} \\ \boldsymbol{\eta}_{12z} \\ \boldsymbol{\eta}_{8z} \\ \boldsymbol{\eta}_{9z} \\ \boldsymbol{\eta}_{15z} \\ \boldsymbol{\eta}_{16z} \end{bmatrix}$$

Simulation

$$\ddot{\boldsymbol{\theta}} = \mathbf{M}^{-1}(\boldsymbol{\theta}) \left[\boldsymbol{\tau} - \mathbf{V}(\boldsymbol{\theta}\ \dot{\boldsymbol{\theta}}) - \mathbf{G}(\boldsymbol{\theta}) - \mathbf{F} \right]$$

initial conditions:

$$\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0 \qquad \dot{\boldsymbol{\theta}}(0) = \ddot{\boldsymbol{\theta}}(0) = 0$$

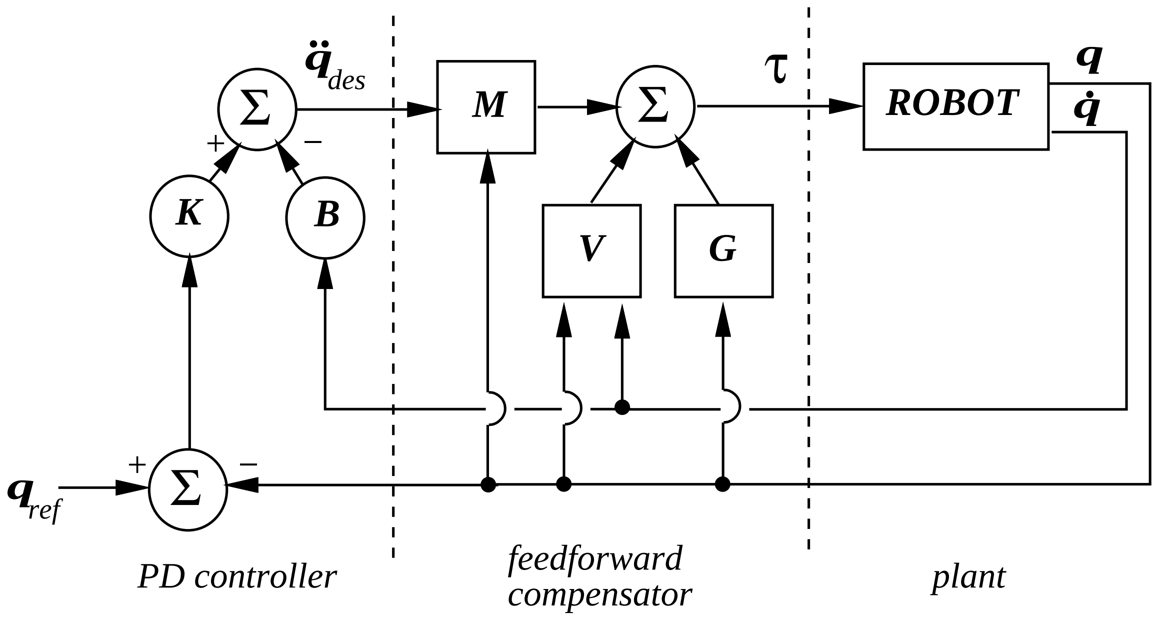
numerical integration:

$$\ddot{\boldsymbol{\theta}}(t) = \mathbf{M}^{-1}[\boldsymbol{\tau} - \mathbf{V} - \mathbf{G} - \mathbf{F}]$$

$$\dot{\boldsymbol{\theta}}(t + \Delta t) = \dot{\boldsymbol{\theta}}(t) + \ddot{\boldsymbol{\theta}}(t)\Delta t$$

$$\boldsymbol{\theta}(t + \Delta t) = \boldsymbol{\theta}(t) + \dot{\boldsymbol{\theta}}(t)\Delta t + \tfrac{1}{2}\ddot{\boldsymbol{\theta}}(t)\Delta t^2$$

Feedforward Dynamic Compensators



linearized and decoupled

Generalized Inertia Ellipsoid

computed torque equation:

$$\boldsymbol{\tau} = \mathbf{M}\ddot{\boldsymbol{\theta}} + \mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta})$$

if we assume that $\dot{\boldsymbol{\theta}} \approx 0$, and we ignore gravity

$$\boldsymbol{\tau} = \mathbf{M}\ddot{\boldsymbol{\theta}}$$

$$\|\ddot{\boldsymbol{\theta}}\| \leq 1$$

relative inertia—torque required to create a unit acceleration defined by the eigenvalues and eigenvectors of $\mathbf{M}\mathbf{M}^T$

Acceleration Polytope

gravity, actuator performance, and the current state of motion influences the ability of a manipulator to generate accelerations

differentiating $\dot{\boldsymbol{r}} = \mathbf{J}\dot{\boldsymbol{q}}$,

$$\begin{aligned} \ddot{\boldsymbol{r}} &= \mathbf{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\mathbf{J}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \\ &= \mathbf{J}\left[\mathbf{M}^{-1}(\boldsymbol{\tau} - \mathbf{V} - \mathbf{G})\right] + \dot{\mathbf{J}}\dot{\boldsymbol{q}} \\ &= \mathbf{J}\mathbf{M}^{-1}\boldsymbol{\tau} + \dot{\boldsymbol{v}}_{vel} + \dot{\boldsymbol{v}}_{grav}, \end{aligned}$$

$$\begin{aligned} \dot{\boldsymbol{v}}_{vel} &= -\mathbf{J}\mathbf{M}^{-1}\mathbf{V} + \dot{\mathbf{J}}\dot{\boldsymbol{q}}, \qquad \text{and} \\ \dot{\boldsymbol{v}}_{grav} &= -\mathbf{J}\mathbf{M}^{-1}\mathbf{G}. \end{aligned}$$

$$\tilde{\boldsymbol{\tau}} = \mathbf{L}^{-1}\boldsymbol{\tau} \qquad \qquad \mathbf{L} = diag({\tau_1}^{limit}, \ldots, {\tau_n}^{limit})$$

admissible torques constitute a unit hypercube $\|\tilde{\boldsymbol{\tau}}\|_\infty \leq 1$

$$\begin{aligned} \ddot{\boldsymbol{r}} &= \mathbf{J}\mathbf{M}^{-1}\mathbf{L}\tilde{\boldsymbol{\tau}} + \dot{\boldsymbol{v}}_{vel} + \dot{\boldsymbol{v}}_{grav} \\ &= \mathbf{J}\mathbf{M}^{-1}\mathbf{L}\tilde{\boldsymbol{\tau}} + \dot{\boldsymbol{v}}_{bias}. \end{aligned}$$

maps the n-dimensional hypercube $\|\tilde{\boldsymbol{\tau}}\|_\infty \leq 1$
to the m-dimensional *acceleration polytope*

Dynamic Manipulability Ellipsoid

$$\tilde{\boldsymbol{\tau}}^T \tilde{\boldsymbol{\tau}} = (\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})^T \left(\left[\mathbf{J}\mathbf{M}^{-1}\mathbf{L} \right]^{-1} \right)^T \left(\left[\mathbf{J}\mathbf{M}^{-1}\mathbf{L} \right]^{-1} \right) (\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias}) \leq 1$$

$\mathbf{M}$ and $\mathbf{L}$ are symmetric:

$\mathbf{A}^{-T} = (\mathbf{A}^{-1})^T$, $\mathbf{A}^{-2} = \mathbf{A}^{-1}\mathbf{A}^{-1}$, and for symmetric matrices, $\mathbf{A}^T = \mathbf{A}$.

$$(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})^T \left[\mathbf{J}^{-T}\mathbf{M}\mathbf{L}^{-2}\mathbf{M}\mathbf{J}^{-1} \right] (\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias}) \leq 1,$$

so that

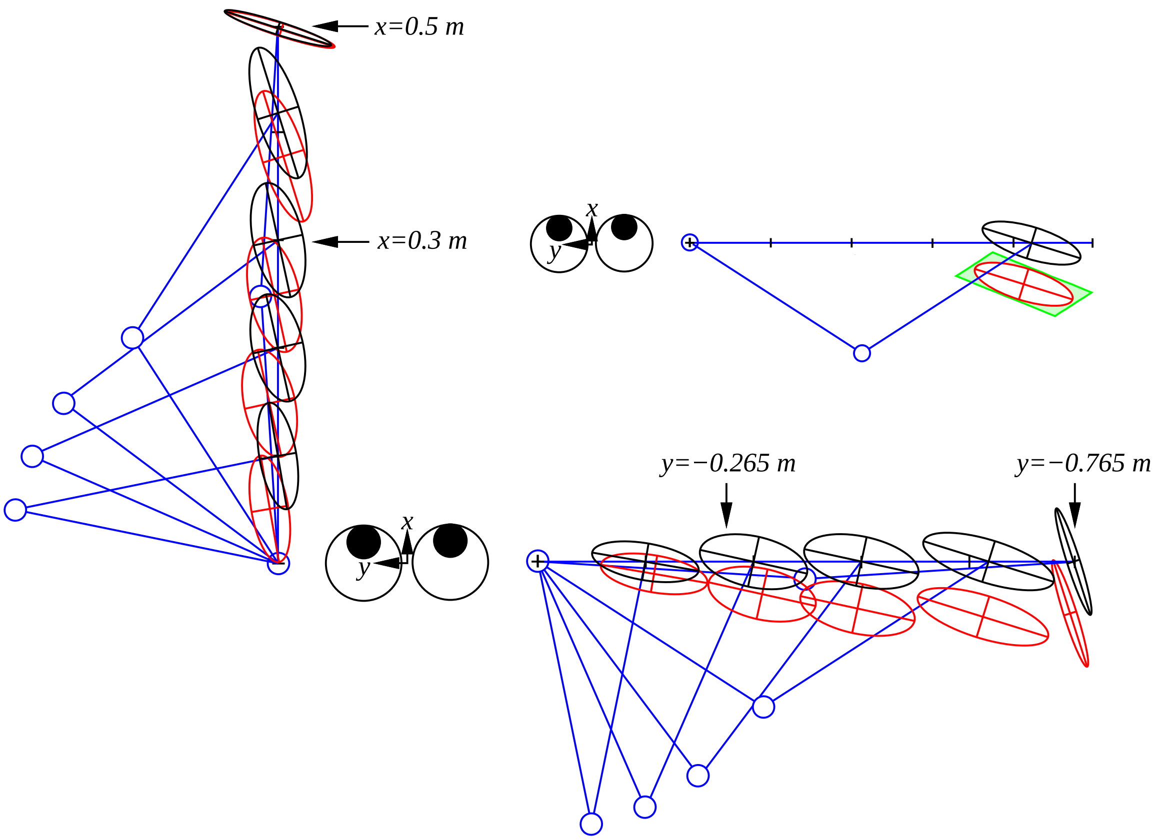
dynamic manipulability ellipsoid

$$(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{v}}_{bias})^T \in \left[\mathbf{J}\mathbf{M}^{-T}\mathbf{L}^2\mathbf{M}^{-1}\mathbf{J}^T \right]$$

dynamic-manipulability measure

$$\boldsymbol{\kappa}_d(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sqrt{det \left[\mathbf{J}(\mathbf{M}^T\mathbf{M})^{-1}\mathbf{J}^T \right]}$$

Conditioning Acceleration



$m_1 = m_2 = 0.2\ kg,\ l_1 = l_2 = 0.25\ m,\ \tau^T\tau \leq 0.005\ N^2m^2.$

black ellipsoids - unbiased dynamic manipulability

gravity biased dynamic manipulability

normalized acceleration polytope with gravity bias