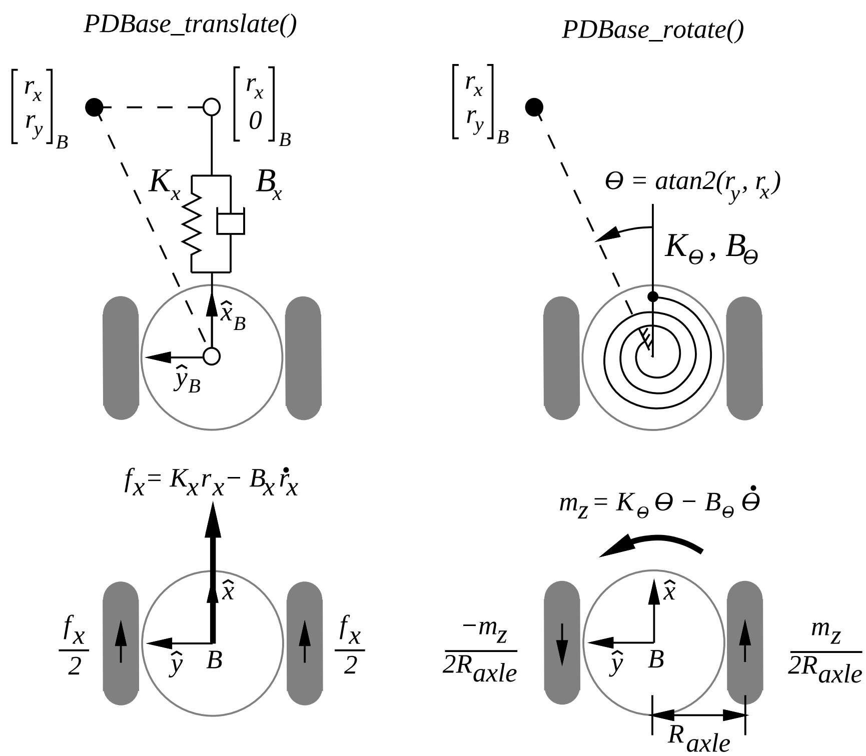
Base Control - Practical Stuff

Two spring-damper configurations with different gains that apply a longitudinal force f_x and rotational moment m_z to the robot.



the GUI provides references in the world coordinate frame

$$\boldsymbol{r}_W = \begin{bmatrix} r_x \\ r_y \end{bmatrix}_W = \begin{bmatrix} x_{ref} - x_{act} \\ y_{ref} - y_{act} \end{bmatrix}_W,$$

BUT, the translation and rotation errors depend on this error written in the base coordinate frame...

Base Control - Practical Stuff

the translation and rotation errors depend on this error written in the base coordinate frame.

$$\bar{\boldsymbol{r}}_B = \begin{bmatrix} r_x \\ r_y \\ 0 \\ 1 \end{bmatrix}_B = {}_BT_W \begin{bmatrix} r_x \\ r_y \\ 0 \\ 1 \end{bmatrix}_W$$

where, ${}_BT_W$ is a 4×4 *homogeneous transform*

and the translation error is just the $\hat{\boldsymbol{x}}$ component of $\bar{\boldsymbol{r}}_B$.

procedure `construct_wTb()` is provided for your use

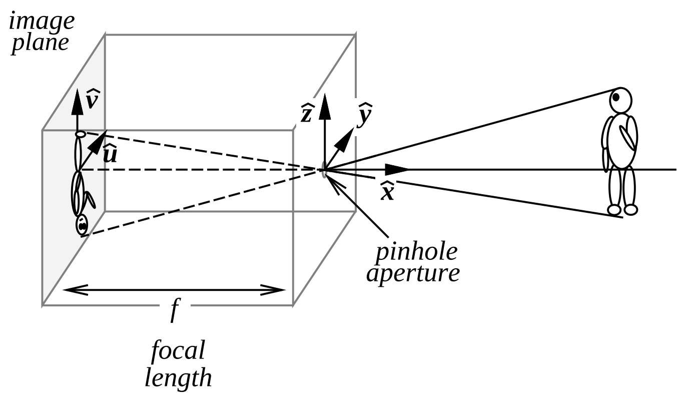
The command $[\ f_x\ m_z\ ]^T$ on the base is transformed into wheel torques using via the differential steering geometry.

$$\boldsymbol{\tau} = \mathbf{J}^T\ \mathbf{w}$$

$$\begin{bmatrix} \tau_L \\ \tau_R \end{bmatrix} = \begin{bmatrix} 1/2 & -1/(2R_{axle}) \\ 1/2 & 1/(2R_{axle}) \end{bmatrix} \begin{bmatrix} f_x \\ m_z \end{bmatrix}$$

Roger Vision (Foveation)
Practical Stuff

Roger's eyes are pinhole RGB cameras with a focal length of 64 pixels that produce a one dimensional image 128 pixels wide.



coordinate (x, y, z) projects to image plane coordinates

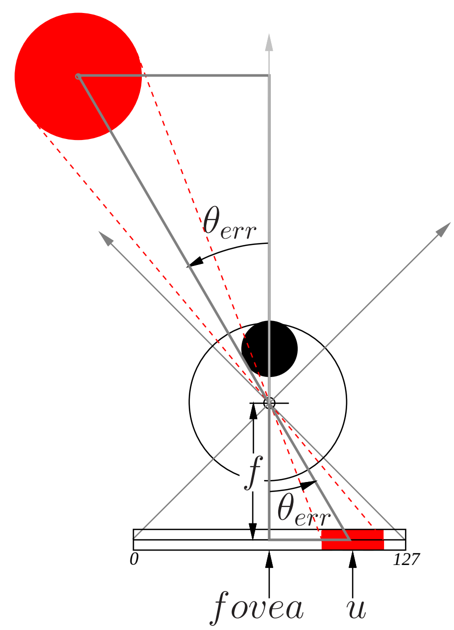
$$\frac{v}{f} = -\frac{z}{x}, \quad \text{and, } \frac{u}{f} = -\frac{y}{x}.$$

Roger Vision (Foveation)
Practical Stuff

implement procedure `average_red_pixel()`: checks for the color *red* on both image planes

```
if (there is red detected in both images) {
   estimate image coordinates, ul and ur,
   of the center of the red segments on the
   left and right images and return(TRUE)
}
else return(FALSE);
```

use *ul* and *ur* to compute the angular error for use in oculomotor controllers that orient each eye to the stimulus—a process called "foveation"—by updating setpoints for the eyes to center the image of the red ball in both image planes.



Kinematics

A branch of dynamics that deals with aspects of motion apart from considerations of force and mass — Websters dictionary

links - individual rigid bodies that collectively form a robot.

joints - connect links in pairs using revolute or prismatic constraints.

- **prismatic joint** - one link moves linearly (as in a slider in a guide link) relative to another.
- **revolute joint** - one link rotates about a center of rotation (a bearing) rigidly connected to another link.

kinematic chain - an assemblage of links connected via joints.

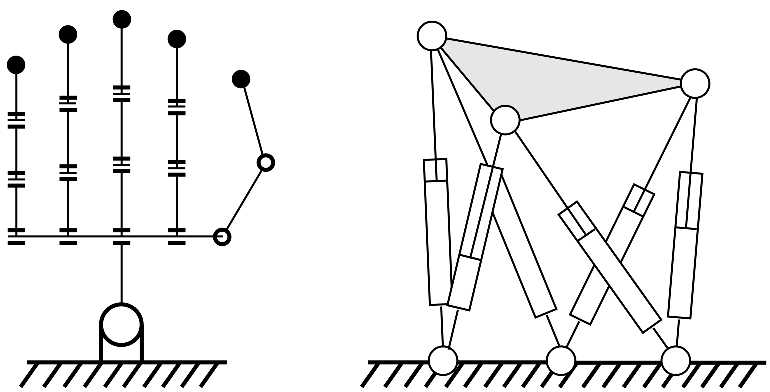
mechanism - a kinematic chain with one fixed (ground) link.

closed chain - a kinematic chain with every link connected through joints to two adjacent links.

Kinematics (cont.)

open chain - a kinematic chain where one link (the unitary link) is connected to a single joint.

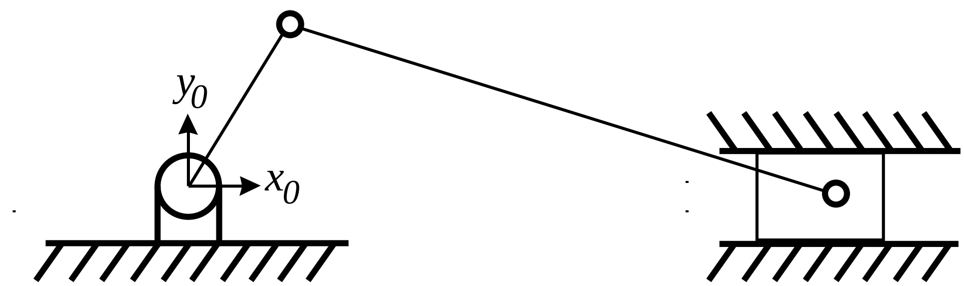
parallel chain - a mechanism with open or closed chains connected through multiple joints to a common link



configuration variables - the parameters (lengths or angles) of a mechanism that can be used to determine the spatial configuration of the mechanism.

degrees of freedom The minimum number of configuration variables necessary to fully define the configuration of a mechanism.

Example - A Familiar Mechanism



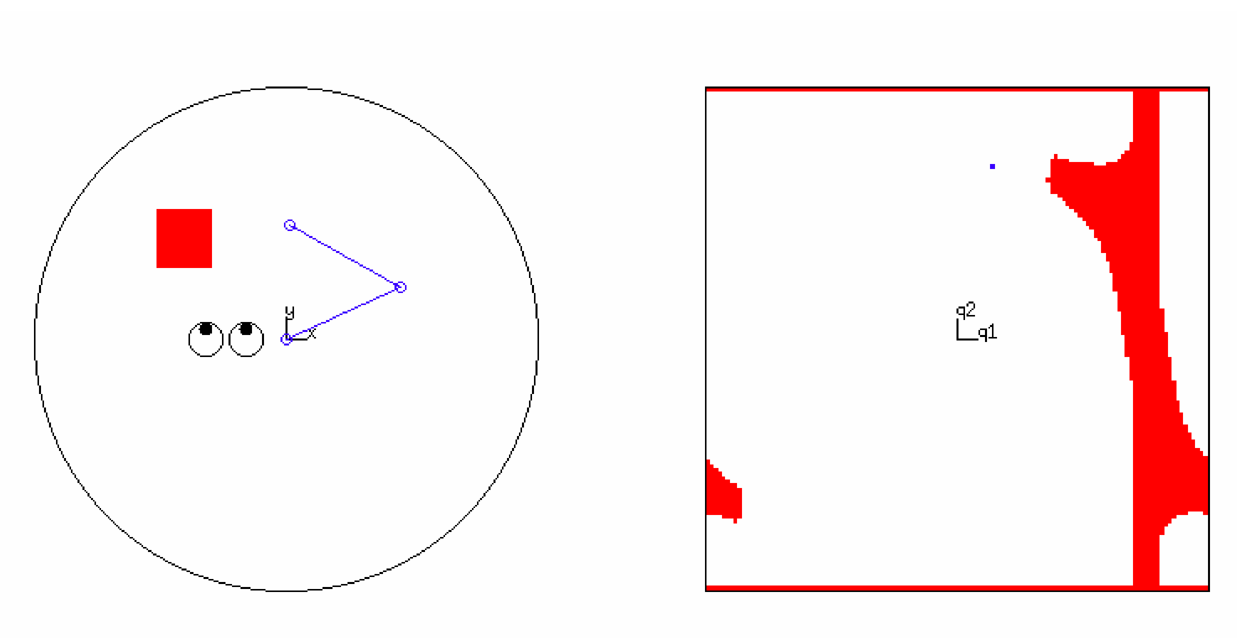
1. how many links does it have?

2. how many joints does it have?

3. how many degrees of freedom does it have?

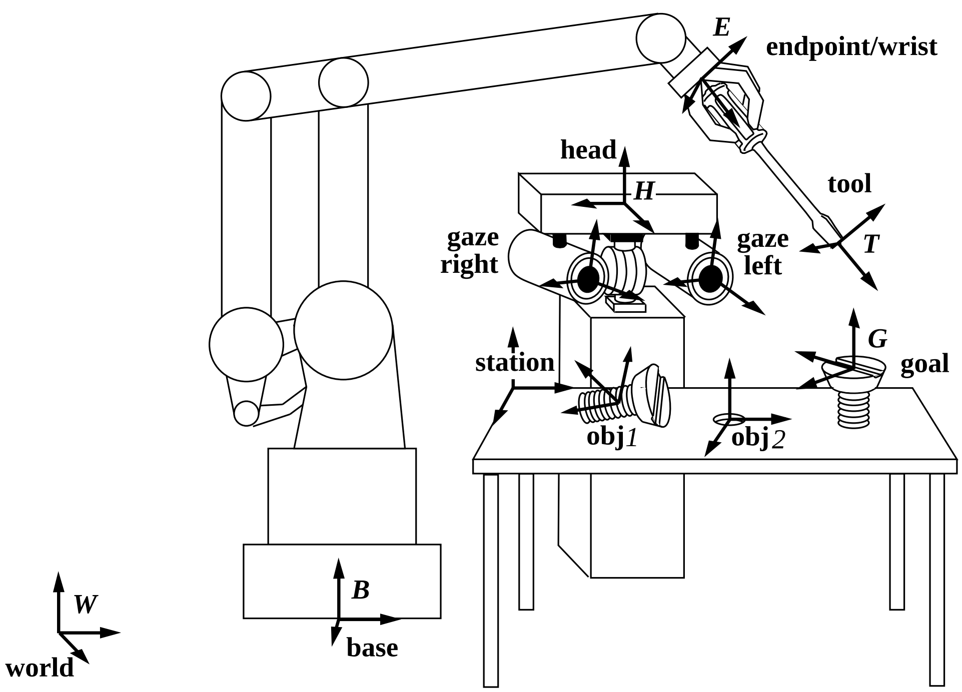
Configuration Space

...the space defined by independently controllable configuration variables in which a particular configuration is a single (point) coordinate

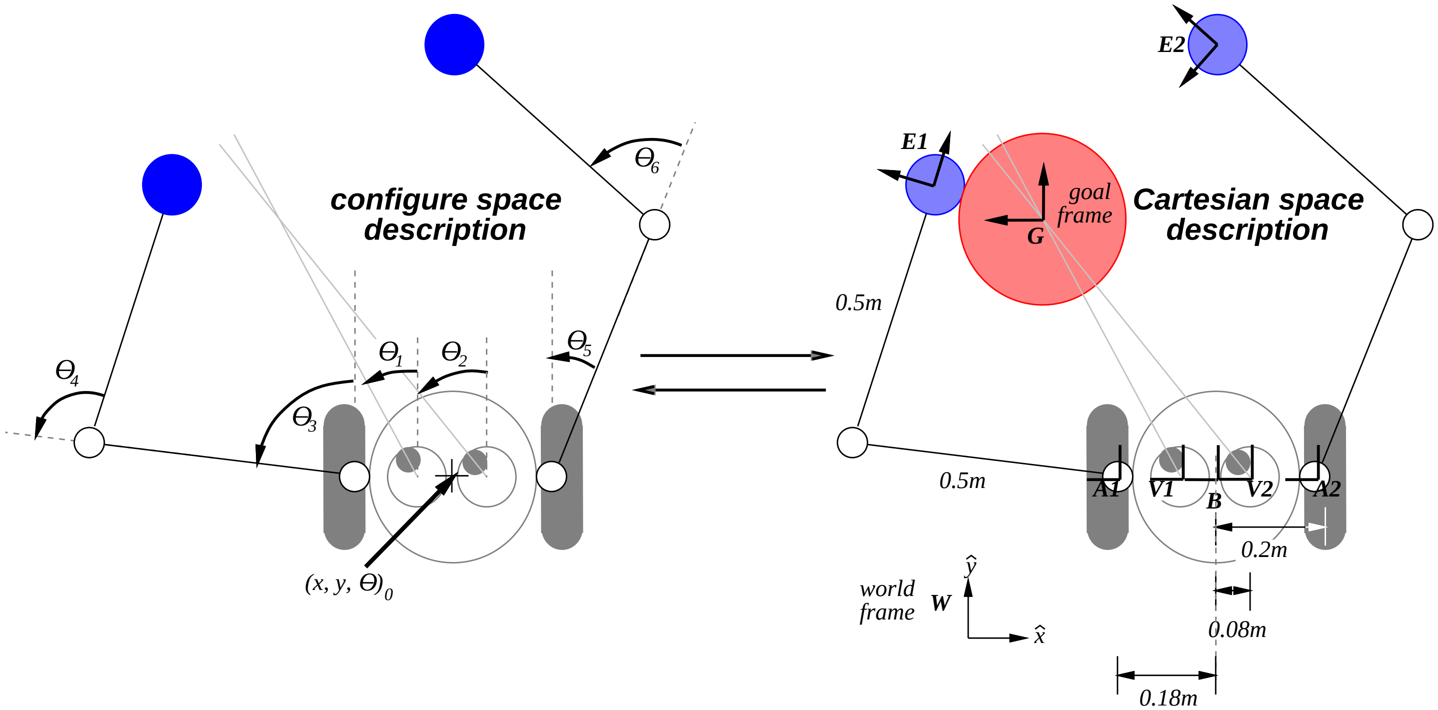


demonstration - C/Roger/harmonic_fnc/x

Spatial Tasks



Two Different Spatial Representations

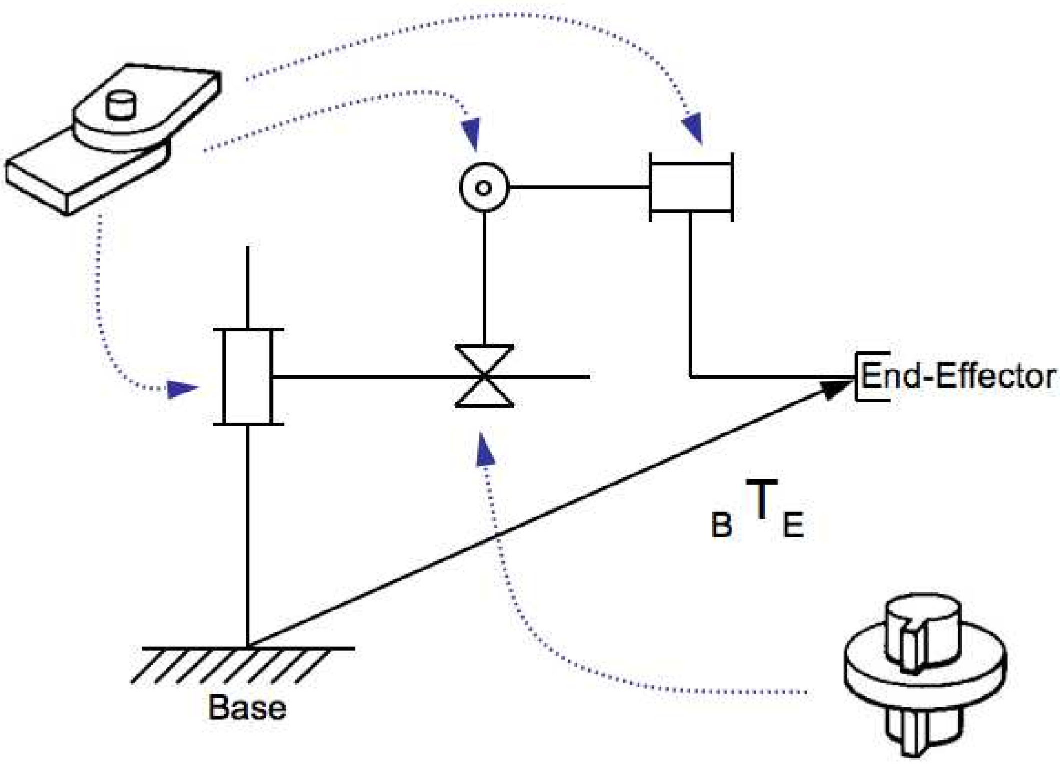


"nonholonomic" contraints:

$$f(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{l}^T \dot{\boldsymbol{q}} = [sin(\theta) \text{ -}cos(\theta) \; 0] \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0,$$

where, $\boldsymbol{l}$ is the vector in the x-y plane that is orthogonal to the current vehicle heading (the lateral direction).

Schematic Diagrams of Open-Chain Mechanisms



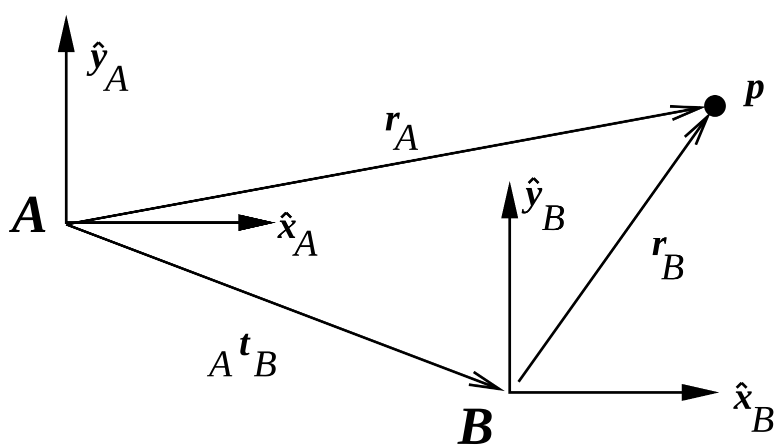
Spatial Relationships

Free Bodies

A *free body* has 6 spatial degrees of freedom:

$$\text{translations: } \boldsymbol{t} \in \mathbb{R}^3$$
$$\text{rotations: } \mathbf{R} \in SO(3)$$

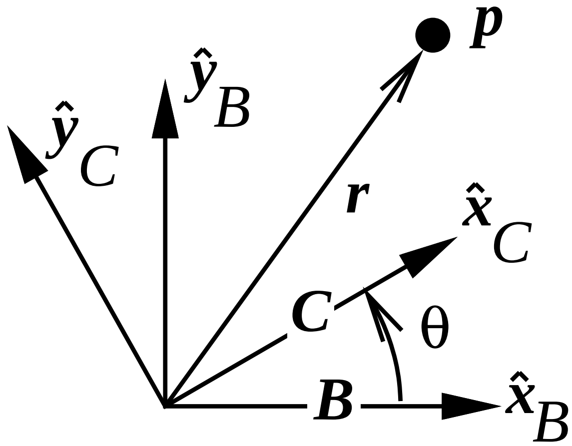
Translation



$$\boldsymbol{r}_A = \boldsymbol{r}_B +_A \boldsymbol{t}_B$$

Spatial Relationships

Rotations



$$\boldsymbol{r}^B = r_x^B \hat{i}_B + r_y^B \hat{j}_B + r_z^B \hat{k}_B$$

$$\boldsymbol{r}^C = r_x^C \hat{i}_C + r_y^C \hat{j}_C + r_z^C \hat{k}_C$$

$$\boldsymbol{r}^B = {}_B\mathbf{R}_C \, \boldsymbol{r}^C$$

$\hat{\boldsymbol{x}}_C^B$, $\hat{\boldsymbol{y}}_C^B$, and $\hat{\boldsymbol{z}}_C^B$ specify the $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ axes of frame C written in frame B coordinates.

$$\begin{matrix} & & \hat{\boldsymbol{x}}_C^B & \hat{\boldsymbol{y}}_C^B & \hat{\boldsymbol{z}}_C^B \end{matrix}$$

$$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}_B = \begin{bmatrix} \hat{i}_B \cdot \hat{i}_C & \hat{i}_B \cdot \hat{j}_C & \hat{i}_B \cdot \hat{k}_C \\ \hat{j}_B \cdot \hat{i}_C & \hat{j}_B \cdot \hat{j}_C & \hat{j}_B \cdot \hat{k}_C \\ \hat{k}_B \cdot \hat{i}_C & \hat{k}_B \cdot \hat{j}_C & \hat{k}_B \cdot \hat{k}_C \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}_C$$

Properties of the Rotation Matrix

- columns and rows of $\mathbf{R}$ are orthonormal (orthogonal and unit length)

- $\mathbf{R}^{-1} = \mathbf{R}^T$

- $det(\mathbf{R}) = +1$ for right-handed convention

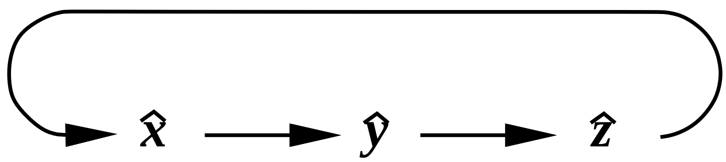
- the set of all $n \times n$ matrices $\mathbf{R}$ that have these properties are called the **Special Orthogonal group** of order n

$$\mathbf{R} \in SO(n)$$

Special Properties of $SO(3)$

- right hand rule:

$$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}$$
$$\hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}$$
$$\hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}$$



right handed coordinate systems

- $SO(3)$ is a "group" under multiplication:

  1. closure: if $\mathbf{R}_1, \mathbf{R}_2 \in SO(3) \Rightarrow \mathbf{R}_1\mathbf{R}_2 \in SO(3)$
  2. identity:

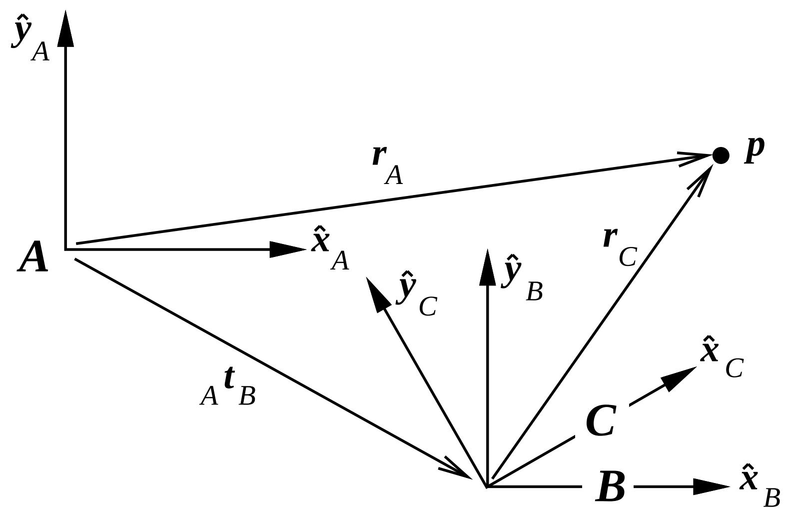
  $$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SO(3)$$

  3. inverse: $\mathbf{R}^{-1} = \mathbf{R}^T$
  4. associativity: $(\mathbf{R}_1\ \mathbf{R}_2)\ \mathbf{R}_3 = \mathbf{R}_1\ (\mathbf{R}_2\ \mathbf{R}_3)$
  5. in general elements of $SO(3)$ do not commute:

  $$\mathbf{R}_1\ \mathbf{R}_2 \neq \mathbf{R}_2\ \mathbf{R}_1$$

Spatial Relationships

The Homogeneous Transform



$${}_A\mathbf{T}_C = \left[\begin{array}{ccc|c} & {}_B\mathbf{R}_C & & {}_A\boldsymbol{t}_B \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \in SE(3) \qquad \boldsymbol{r}_C = \begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix}_C \in \mathbb{R}^3$$

$$\begin{aligned} \boldsymbol{r}_A &= {}_A\mathbf{T}_C\, \boldsymbol{r}_C \\ &= {}_B\mathbf{R}_C\, \boldsymbol{r}_C + {}_A\boldsymbol{t}_B \end{aligned}$$

the "1" creates the *homogeneous position vector*

The Homogeneous Transform

$$ {}_{A}T_{C} = \left[\begin{array}{ccc|c} & {}_{B}R_{C} & & {}_{A}\boldsymbol{t}_{B} \\ & & & \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \qquad \boldsymbol{r}_{A} = {}_{A}T_{C}\, \boldsymbol{r}_{C} $$

- translate *then* rotate
- indicial notation:
  - the *sign* of the transform is determined left to right, i.e. A to C defines the sign of the rotation
  - it transforms homogeneous position vectors in frame C into homogeneous position vectors in frame A

The Homogeneous Transform

$$trans(\boldsymbol{t}) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$rot(\hat{\boldsymbol{x}}, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) & 0 \\ 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$rot(\hat{\boldsymbol{y}}, \theta) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta) & 0 & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$rot(\hat{\boldsymbol{z}}, \theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

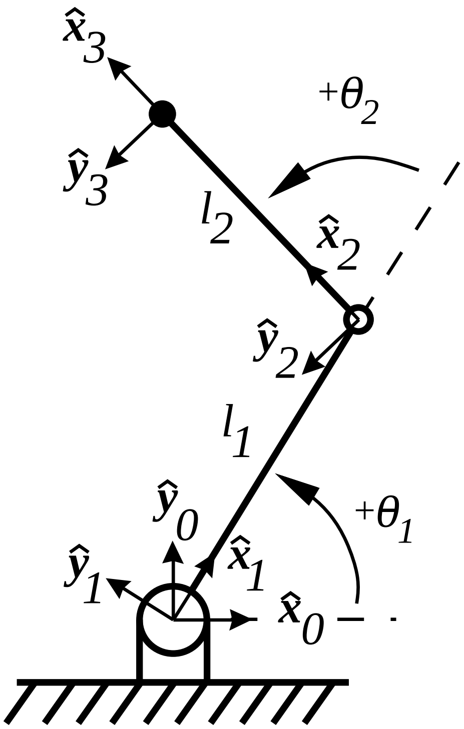
Spatial Relationships

Inverting the Homogeneous Transform

$$
{}_A\mathbf{T}_C = \begin{bmatrix} \hat{\boldsymbol{x}}_C^B & \hat{\boldsymbol{y}}_C^B & \hat{\boldsymbol{z}}_C^B & {}_A\boldsymbol{t}_B \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}_C\mathbf{T}_A = [\ {}_A\mathbf{T}_C]^{-1} = \begin{bmatrix} & (\hat{\boldsymbol{x}}_C^B)^T & & (-\boldsymbol{t}\cdot\hat{\boldsymbol{x}}_C^B) \\ & (\hat{\boldsymbol{y}}_C^B)^T & & (-\boldsymbol{t}\cdot\hat{\boldsymbol{y}}_C^B) \\ & (\hat{\boldsymbol{z}}_C^B)^T & & (-\boldsymbol{t}\cdot\hat{\boldsymbol{z}}_C^B) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Forward Kinematics: EXAMPLE



$$
\begin{aligned}
x &= l_1 cos(\theta_1) + l_2 cos(\theta_1 + \theta_2) \\
y &= l_1 sin(\theta_1) + l_2 sin(\theta_1 + \theta_2)) \\
\theta &= \theta_1 + \theta_2
\end{aligned}
$$

$$
{}_0T_3 = {}_0T_1 \; {}_1T_2 \; {}_2T_3
$$

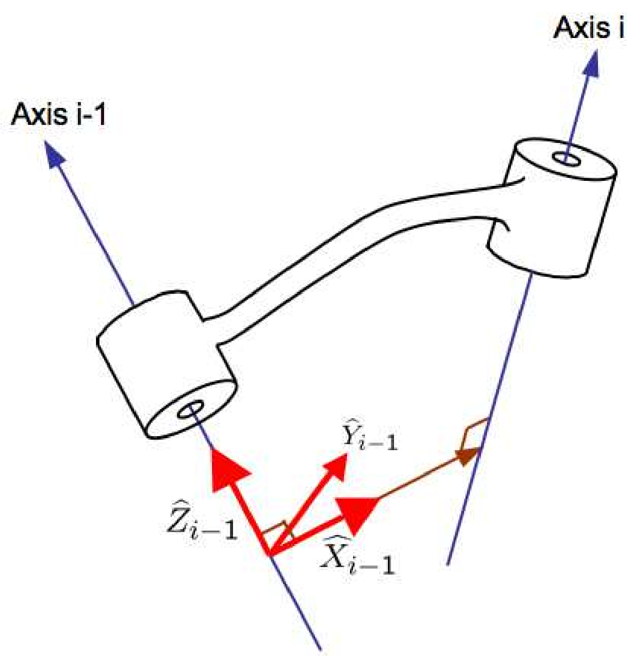
$$
= \begin{bmatrix} cos_{12} & -sin_{12} & 0 & l_1 cos_1 + l_2 cos_{12} \\ sin_{12} & cos_{12} & 0 & l_1 sin_1 + l_2 sin_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Useful Trignometric Identities

$$
\begin{aligned}
cos(\theta_1 + \theta_2) &= cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) \\
cos(\theta_1 - \theta_2) &= cos(\theta_1)cos(\theta_2) + sin(\theta_1)sin(\theta_2) \\
sin(\theta_1 + \theta_2) &= sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2) \\
sin(\theta_1 - \theta_2) &= sin(\theta_1)cos(\theta_2) - cos(\theta_1)sin(\theta_2) \\
sin(\theta_2) &= cos(\theta_1)sin(\theta_1 + \theta_2) - sin(\theta_1)cos(\theta_1 + \theta_2) \\
cos(\theta_2) &= cos(\theta_1)cos(\theta_1 + \theta_2) + sin(\theta_1)sin(\theta_1 + \theta_2)
\end{aligned}
$$

The Denavit-Hartenberg Conventions

Coordinate Frames



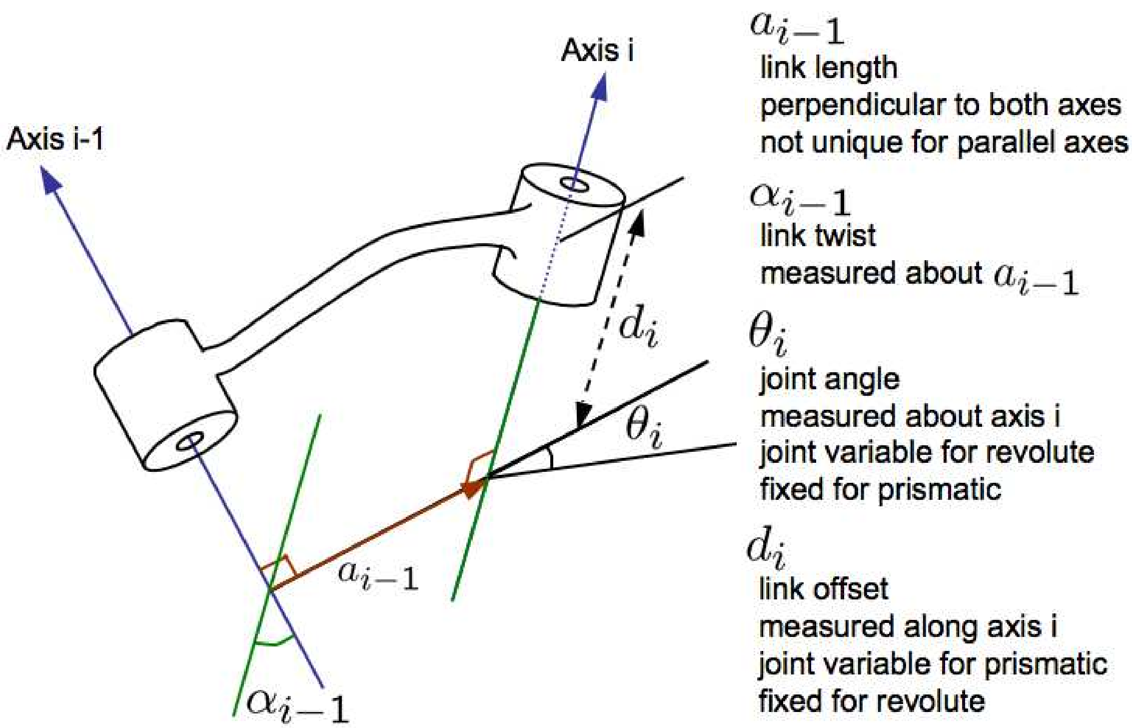
$\widehat{Z}_{i-1}$
along joint axis of joint **i-1**

$\widehat{X}_{i-1}$
along perpendicular from joint axis **i-1** to joint axis **i** (note special case for intersecting axes)

$\widehat{Y}_{i-1}$
results from right-hand-rule

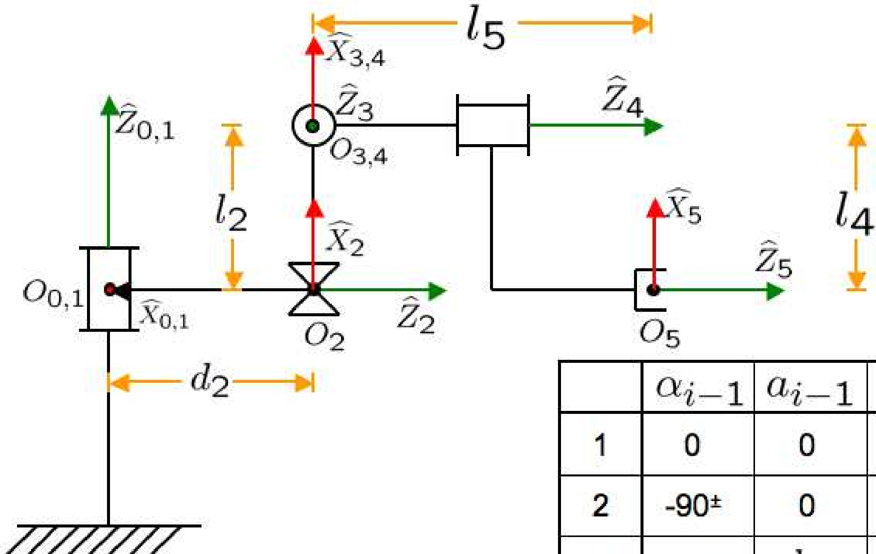
Denavit-Hartenberg Parameters

from $frame_{i-1}$ to $frame_i$:



3 fixed parameters per joint, one variable

Denavit-Hartenberg - 4 DOF example



	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90±	0	d_2	-90±
3	-90±	l_2	0	θ_3
4	90±	0	0	θ_4
5	0	$-l_4$	l_5	0

Denavit-Hartenberg - Parametric Homogeneous Transform

α_i = the angle between $\hat{Z}_i$ and $\hat{Z}_{i+1}$ measured about $\widehat{X}_i$

a_i = the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\widehat{X}_i$

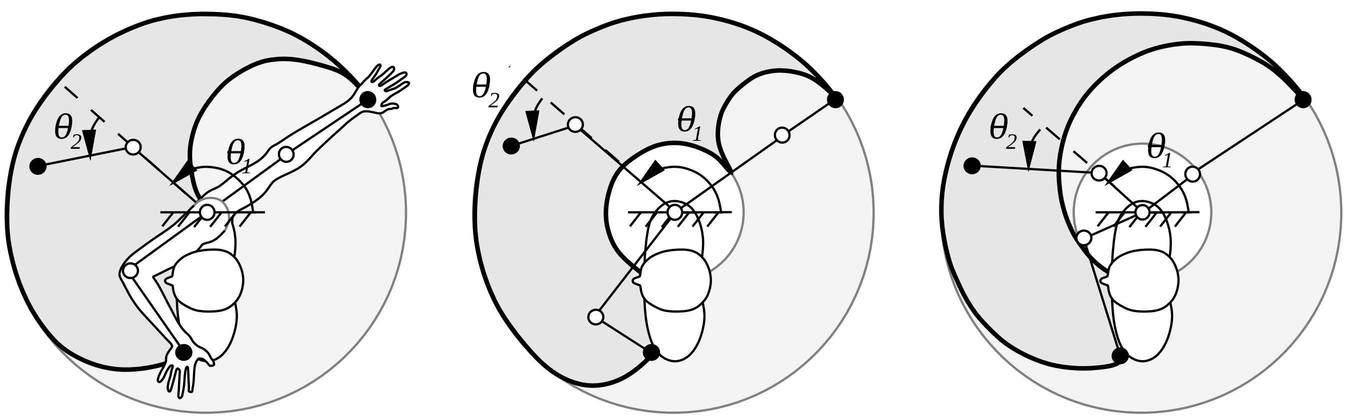
d_i = the distance from $\widehat{X}_{i-1}$ to $\widehat{X}_i$ measured along $\hat{Z}_i$

θ_i = the angle between $\widehat{X}_{i-1}$ and $\widehat{X}_i$ measured about $\hat{Z}_i$

$$
\begin{aligned}
{}_{i\text{-}1}T_i &= R_X(\alpha_{i-1})\, D_X(a_{i-1})\, R_Z(\theta_i)\, D_Z(d_i) \\
&= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Inverse Kinematics: $X \mapsto \Theta$

reachability, dexterity, multiple solutions

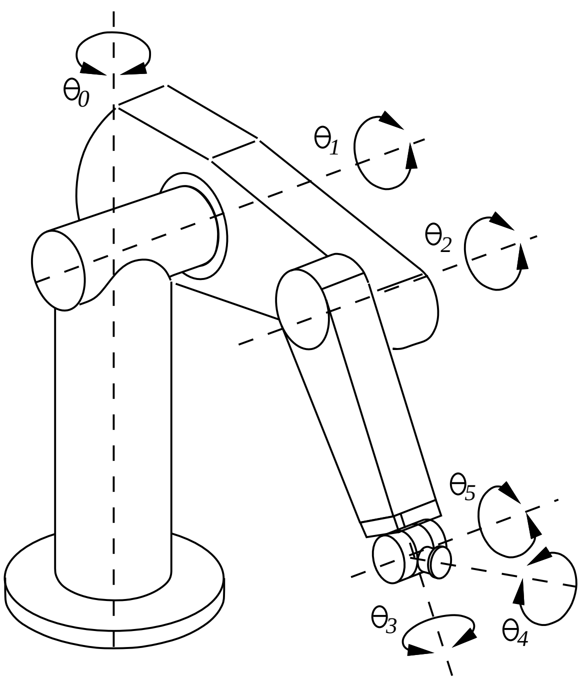


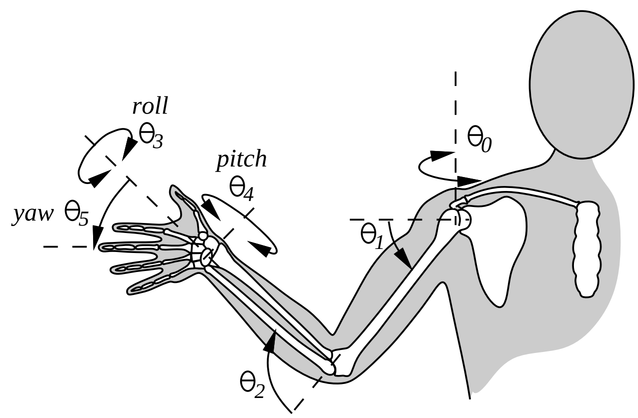
$l_1 = l_2$ $l_1 > l_2$ $l_2 > l_1$

Closed-Form Inverse Kinematic Solutions

- Pieper (ca. 1968) general inverse kinematic solution

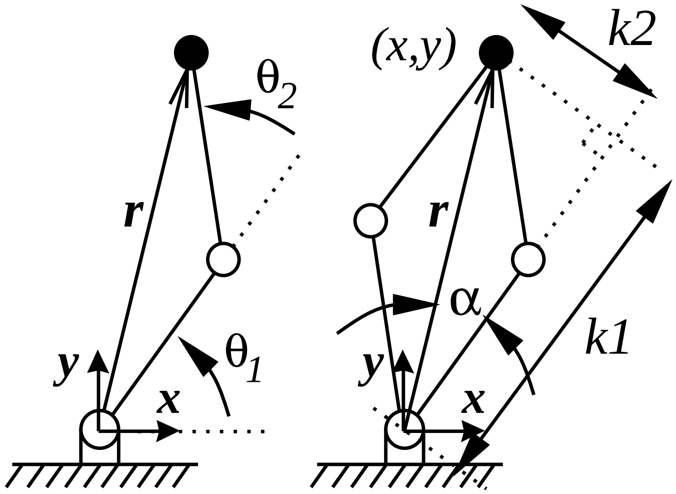
  6 revolute joints have a closed form solution if 3 neighboring joint axes intersect at a point





- Paul (ca. 1981) homogeneous transform-based generalized IK
- Geometric Techniques

Inverse Kinematics: EXAMPLE



- eliminate θ_1, solve for two unique θ_2 solutions:

$$
\begin{aligned}
r^2 &= x^2 + y^2, \; and \\
x &= l_1 c_1 + l_2 c_{12} \\
y &= l_1 s_1 + l_2 s_{12}
\end{aligned}
$$

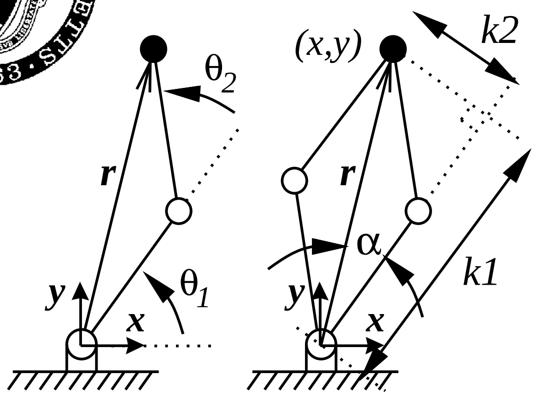
$$
\begin{aligned}
s_{12} &= s_1 c_2 + c_1 s_2 \\
c_{12} &= c_1 c_2 - s_1 s_2
\end{aligned}
$$

$$
\begin{aligned}
r^2 = x^2 + y^2 &= l_1^2 c_1^2 + 2 l_1 l_2 c_1 c_{12} + l_2^2 c_{12}^2 \\
&\quad + l_1^2 s_1^2 + 2 l_1 l_2 s_1 s_{12} + l_2^2 s_{12}^2 \\
&= l_1^2 + 2 l_1 l_2 c_2 + l_2^2
\end{aligned}
$$

and,

$$c_2 = \frac{r^2 - l_1^2 - l_2^2}{2 l_1 l_2}, \; c_2 \in [-1, +1]$$

Inverse Kinematics: EXAMPLE



$$k_1 = rc_\alpha = l_1 + l_2 c_2$$
$$k_2^{+/-} = rs_\alpha = l_2 s_2^{+/-}$$

solve for both θ_2 solutions

$$\begin{aligned} s_2^2 + c_2^2 &= 1 \\ s_2^2 &= 1 - c_2^2 \\ s_2^{+/-} &= +/-(1 - c_2^2)^{1/2} \end{aligned}$$

$$\theta_2^{+/-} = tan^{-1}\frac{s_2^{+/-}}{c_2}, \quad and$$
$$\alpha^{+/-} = tan^{-1}\frac{k_2^{+/-}}{k_1}$$

Therefore,

$$x = k_1 c_1 + k_2 s_1 = (rc_\alpha)c_1 + (rs_\alpha)s_1 = rcos(\alpha + \theta_1)$$
$$y = k_1 s_1 + k_2 c_1 = (rc_\alpha)s_1 + (rs_\alpha)c_1 = rsin(\alpha + \theta_1)$$

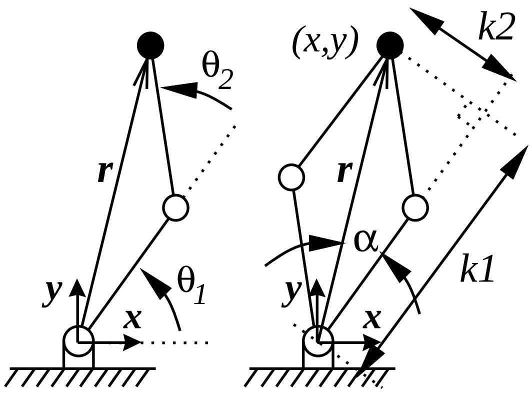
and

$$tan(\alpha + \theta_1) = \frac{rsin(\alpha + \theta_1)}{rcos(\alpha + \theta_1)} = \frac{y}{x}$$

so that,

$$\theta_1^{+/-} = tan^{-1}\frac{y}{x} - \alpha^{+/-}.$$

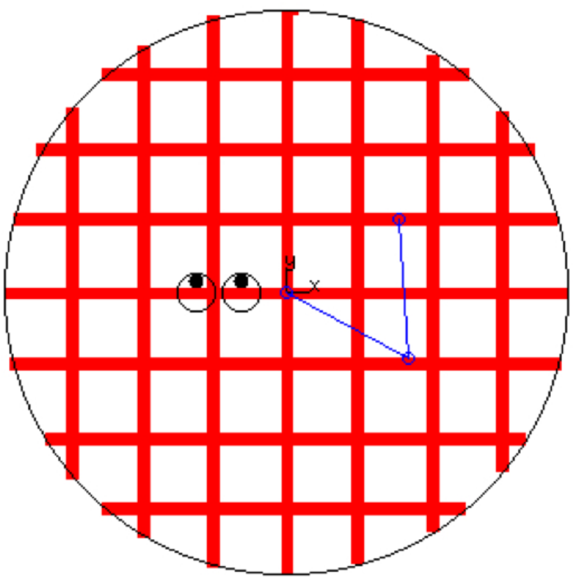
Inverse Kinematics: EXAMPLE

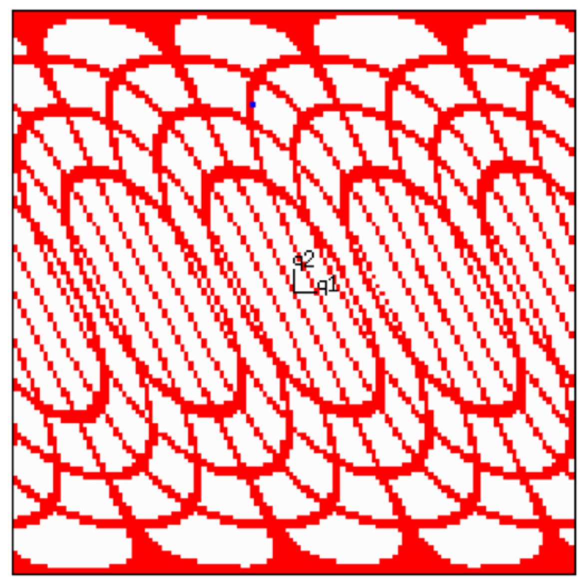


GIVEN (x,y) endpoint position goal:

$$
\begin{aligned}
& r^2 = x^2 + y^2 \\
& c_2 = (r^2 - l_1^2 - l_2^2)/(2 l_1 l_2) \\
& \text{if } (-1 \leq c_2 \leq +1) \\
& \qquad s_2^{+/-} = +/-\,(1 - c_2^2)^{1/2} \\
& \qquad \theta_2^{+/-} = tan^{-1} \tfrac{s_2^{+/-}}{c_2} \\
& \qquad k_1 = l_1 + l_2 c_2 \\
& \qquad k_2^{+/-} = l_2 s_2^{+/-} \\
& \qquad \alpha^{+/-} = tan^{-1} \tfrac{k_2^{+/-}}{k1} \\
& \qquad \theta_1^{+/-} = tan^{-1} \tfrac{y}{x} - \alpha^{+/-} \\
& \text{else “out of reach”}
\end{aligned}
$$

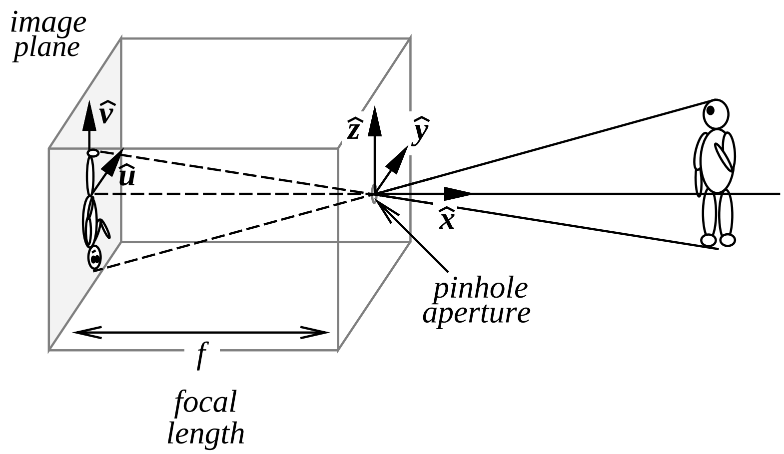
Inverse Kinematics: EXAMPLE

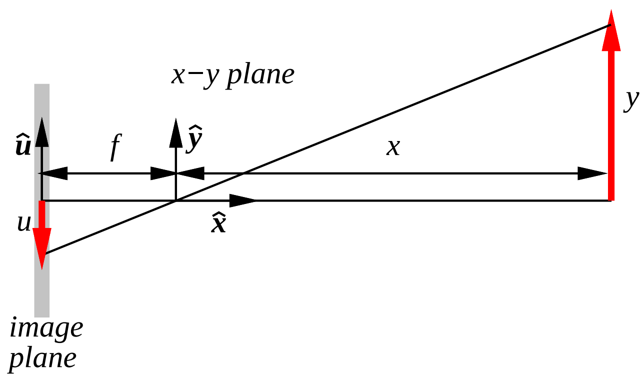


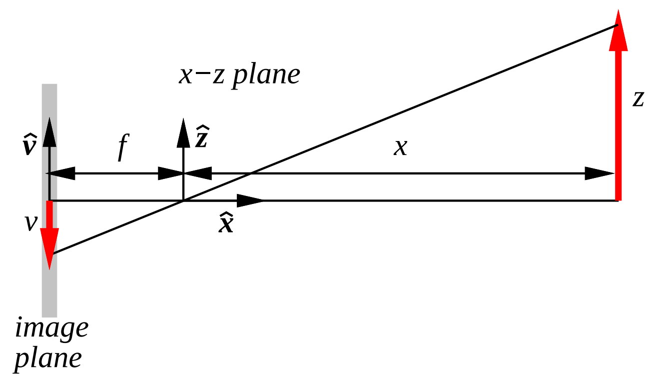


Pinhole Camera

...another kinematic system



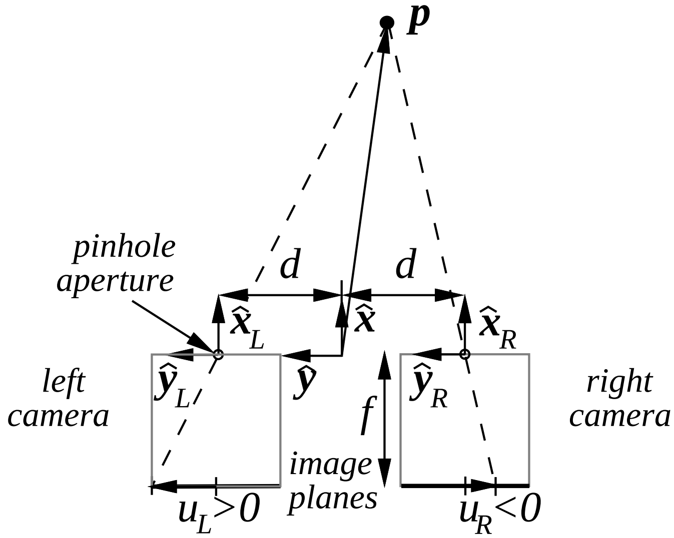




$$u = \frac{-fy}{x} \qquad\qquad v = \frac{-fz}{x}$$

perspective distortion - the pinhole projection distorts Euclidean geometry so that parallel lines converge at “vanishing points.”

Depth Encoded as Disparity



parallel gaze stereo encodes depth entirely in terms of disparity.

$$u_L = \frac{-f(y-d)}{x}$$

$$u_R = \frac{-f(y+d)}{x}$$

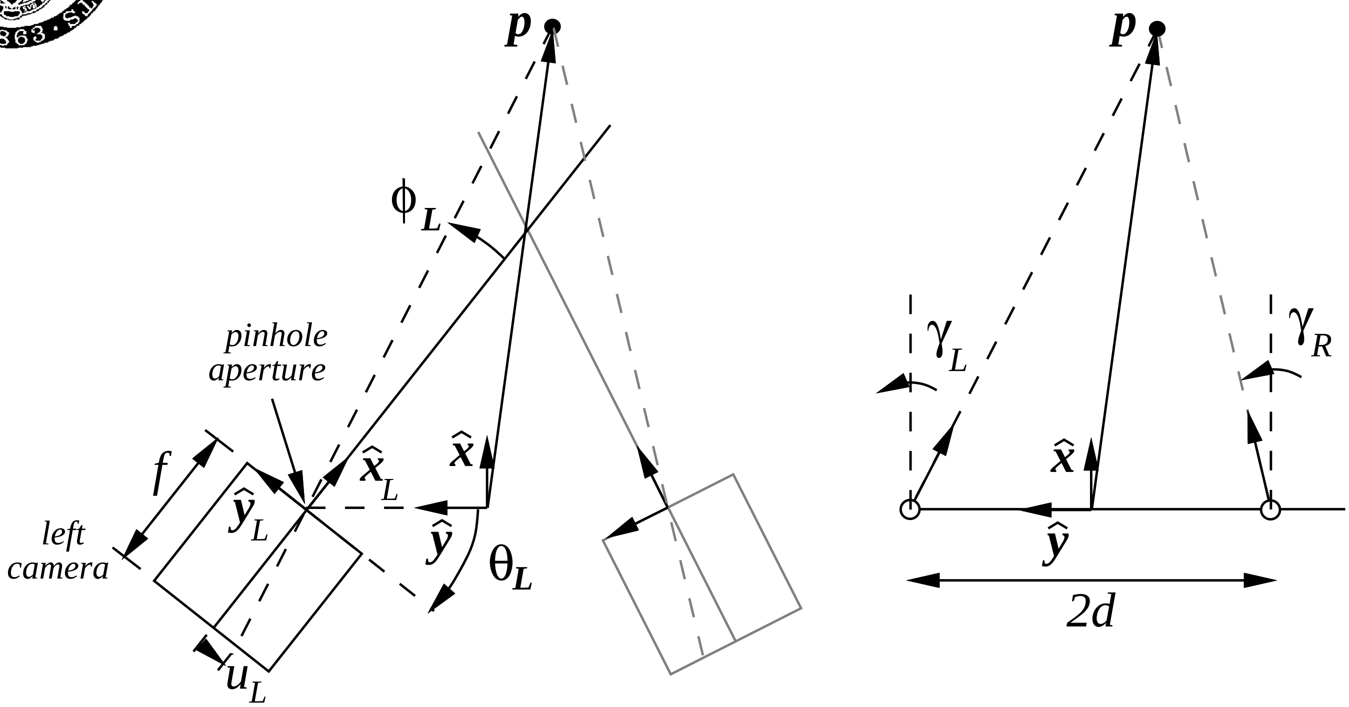
$$\begin{aligned} xu_R &= -f(y+d) \\ xu_L &= -f(y-d) \end{aligned}$$

$$x(u_L - u_R) = 2df$$

and

$$x = \frac{2df}{(u_L - u_R)}$$

Reconstructing Space - Binocular Stereopsis



$$x \ : \ \lambda_L cos(\gamma_L) = \lambda_R cos(\gamma_R),$$

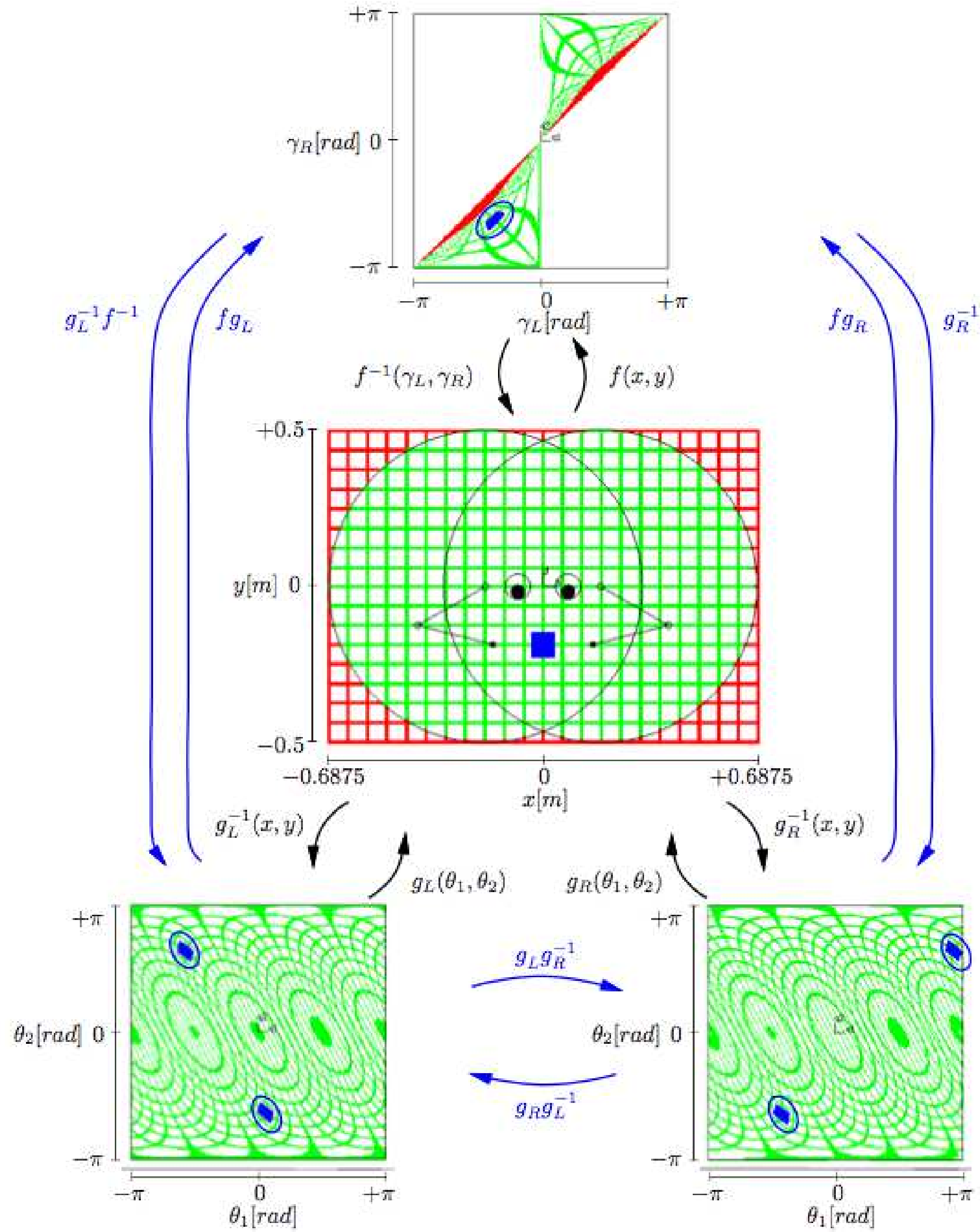
$$\Rightarrow \lambda_L = \lambda_R \frac{cos(\gamma_R)}{cos(\gamma_L)}$$

$$y \ : \ d + \lambda_L sin(\gamma_L) = -d + \lambda_R sin(\gamma_R)$$

$$\lambda_R = \frac{2d cos(\gamma_L)}{sin(\gamma_R - \gamma_L)}, \ \ and \quad \lambda_L = \frac{2d cos(\gamma_R)}{sin(\gamma_R - \gamma_L)}$$

...depth by **vergence** and **disparity**...

Summary - Hand-Eye Spatial Transformations



Jacobian - Locally Linear Kinematic Transformations

nonlinear forward kinematic mapping

$$\boldsymbol{x} = f(\boldsymbol{q})$$

- manipulator forward kinematics
- stereo triangulation equations

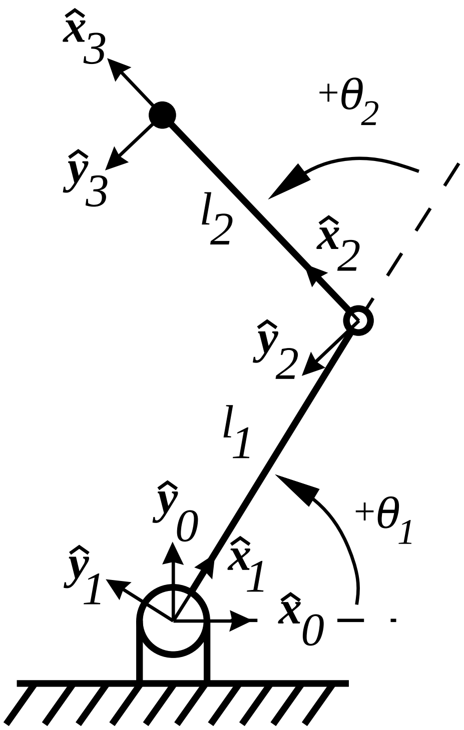
the *differential geometry* of $f(\boldsymbol{q})$ in the neighborhood of $\boldsymbol{q} = \boldsymbol{a}$ is revealed in the Taylor series expansion:

$$f(\boldsymbol{a} + d\boldsymbol{q}) = f(\boldsymbol{a}) + \frac{d\boldsymbol{q}}{1!}\frac{\partial f}{\partial \boldsymbol{q}} + \frac{d\boldsymbol{q}^2}{2!}\frac{\partial^2 f}{\partial \boldsymbol{q}^2} + \ldots, \text{ so that}$$

$$f_a(d\boldsymbol{q}) = df \approx \frac{\partial f}{\partial \boldsymbol{q}} d\boldsymbol{q}$$

$\partial f / \partial \boldsymbol{q}$ is the Jacobian - *a (hyper)plane whose slope is identical to the tangent to the function* $f(\boldsymbol{q})$ *at* $\boldsymbol{q} = \boldsymbol{a}$

The Manipulator Jacobian: linear mapping $d\Theta \mapsto dX$

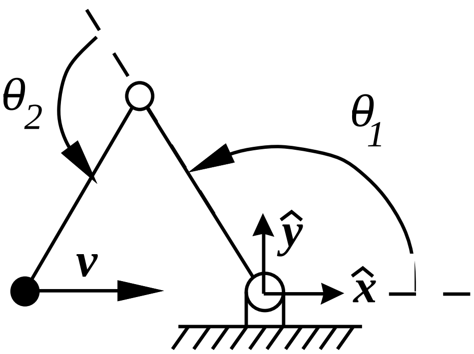


$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}$$

The Inverse Manipulator Jacobian: $d\Theta = J^{-1}dX$



$$det \begin{vmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} \\ l_1c_1 + l_2c_{12} & l_2c_{12} \end{vmatrix}$$

$$\begin{aligned} &= l_1l_2(c_1s_{12} - s_1c_{12}) \\ &= l_1l_2s_2 \end{aligned}$$

The Jacobian is singular when $sin(\theta_2) = 0, \theta_2 = 0, \pi.$ for $V = 1\frac{m}{sec}$

in the $\boldsymbol{x}$ direction

$$[J]^{-1} = \frac{1}{l_1l_2s_2}\begin{bmatrix} l_2c_{12} & l_2s_{12} \\ -l_1c_1 - l_2c_{12} & -l_1s_1 - l_2s_{12} \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &= \frac{1}{l_1l_2s_2}\begin{bmatrix} l_2c_{12} & l_2s_{12} \\ -l_1c_1 - l_2c_{12} & -l_1s_1 - l_2s_{12} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{l_1l_2s_2}\begin{bmatrix} l_2c_{12} \\ -l_1c_1 - l_2c_{12} \end{bmatrix} \end{aligned}$$

The Manipulator Jacobian in the Force Domain

Work out = Work in

$$F^T \Delta x = \tau^T \Delta\theta$$

$\Delta x = J \Delta\theta$, therefore

$$F^T [J \Delta\theta] = \tau^T \Delta\theta$$

$F^T J = \tau^T$, or

$$\tau = J^T F$$

Review - Eigenvalues and Eigenvectors

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x}$$

if A is a real $n \times n$ matrix, the polynomial

$$p(\lambda) = det(\mathbf{A} - \lambda\mathbf{I})$$

is the *characteristic polynomial* of $\mathbf{A}$.

for a root of the characteristic polynomial, λ^*,

$$[\mathbf{A} - \lambda^*\mathbf{I}]\ \boldsymbol{x}^* = \mathbf{0}, \quad \boldsymbol{x}^* \neq \mathbf{0}$$

$p(\lambda)$: roots λ^* are **eigenvalues**

$\boldsymbol{x}^*$ are the **eigenvectors**

Review — EXAMPLE

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Suppose:

$$\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \qquad (\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} (4-\lambda) & -1 \\ 2 & (1-\lambda) \end{bmatrix}$$

$$\begin{aligned} det(\mathbf{A} - \lambda\mathbf{I}) &= (4-\lambda)(1-\lambda) - (2)(-1) \\ &= \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) \end{aligned}$$

so that, $\lambda^*_{1,2} = 2,\ 3$

Review — EXAMPLE

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \lambda_{1,2} = 2,\ 3$$

$$\text{for } \lambda_1 = 2 \Rightarrow (\mathbf{A} - \lambda\mathbf{I}) : \quad \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \boldsymbol{x}_1^* = \mathbf{0}$$

$$2x_1 - x_2 = 0 \quad \Rightarrow x_1 = \frac{x_2}{2} \quad \Rightarrow \boldsymbol{x}_1^* = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\text{for } \lambda_2 = 3 \Rightarrow (\mathbf{A} - \lambda\mathbf{I}) : \quad \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \boldsymbol{x}_2^* = \mathbf{0}$$

$$x_1 - x_2 = 0 \quad \Rightarrow x_1 = x_2 \quad \Rightarrow \boldsymbol{x}_2^* = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

in general, eigenvectors are not necessarily orthogonal!

Review — Quadratic Forms

Consider the quadratic $\mathbf{A}\mathbf{A}^T$:

$$\mathbf{M} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 7 \\ 7 & 5 \end{bmatrix}$$

In this case:

$$\lambda^2 - 22\lambda + 36 = 0 \quad \Rightarrow \lambda_{1,2} = 20.22, \; 1.78$$

when $\lambda_1 = 20.22$:

$$\hat{\boldsymbol{e}}_1 = \begin{bmatrix} 0.9085 \\ 0.4179 \end{bmatrix}$$

when $\lambda_1 = 1.78$:

$$\hat{\boldsymbol{e}}_2 = \begin{bmatrix} -0.4179 \\ 0.9085 \end{bmatrix}$$

the eigenvectors of the positive definite, symmetric quadratic form are always orthogonal

Review — Quadratic Forms

quadratic form can be used to define an ellipsoidal set:

$$\mathcal{E} = \{ \, \boldsymbol{y} \mid \boldsymbol{y}^T \mathbf{M} \boldsymbol{y} \leq k \} \text{ in } \mathbb{R}^n$$

for positive definite, symmetric matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$.

this is easy to see in two dimensions:

$$[\, y_1 \; y_2 \,] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = a y_1^2 + 2b y_1 y_2 + c y_2^2.$$

the eigenvalues and eigenvectors of $\mathbf{M}$ determine the shape and orientation of the ellipse determined by $\boldsymbol{y}^T \mathbf{M} \boldsymbol{y}$.

EXAMPLE: Ellipsoidal Sets from Quadratic Forms

returning to our previous example:

$$\mathbf{M} = \mathbf{A}\mathbf{A}^T = \begin{bmatrix} 17 & 7 \\ 7 & 5 \end{bmatrix} \qquad \begin{array}{cc} \lambda_1 = 20.22 : & \lambda_2 = 1.78 : \\ \hat{\boldsymbol{e}}_1 = \begin{bmatrix} 0.9085 \\ 0.4179 \end{bmatrix} & \hat{\boldsymbol{e}}_2 = \begin{bmatrix} -0.4179 \\ 0.9085 \end{bmatrix} \end{array}$$

$$\mathcal{E} = \{ \ \boldsymbol{y} \mid \boldsymbol{y}^T \mathbf{M} \boldsymbol{y} \leq k \} \text{ in } \mathbb{R}^2$$

The diagonalized form represents the quadratic in the eigenvector basis where $\boldsymbol{y} = e_1 \hat{\boldsymbol{e}}_1 + e_2 \hat{\boldsymbol{e}}_2$, where $\boldsymbol{e} = [e_1 \ e_2]^T$.

$$\boldsymbol{y}^T \begin{bmatrix} 17 & 7 \\ 7 & 5 \end{bmatrix} \boldsymbol{y} = \boldsymbol{e}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \boldsymbol{e} \leq k$$

The boundary of set is defined by the equality $\boldsymbol{y} = \lambda_1 e_1^2 + \lambda_2 e_2^2 = k$.

$$\text{when } e_2 = 0, \ \ \lambda_1 e_1^2 = k, \ \text{ and } e_1 = \sqrt{k/\lambda_1}$$

$$\text{when } e_1 = 0, \ \ \lambda_2 e_2^2 = k, \ \text{ and } e_2 = \sqrt{k/\lambda_2}.$$

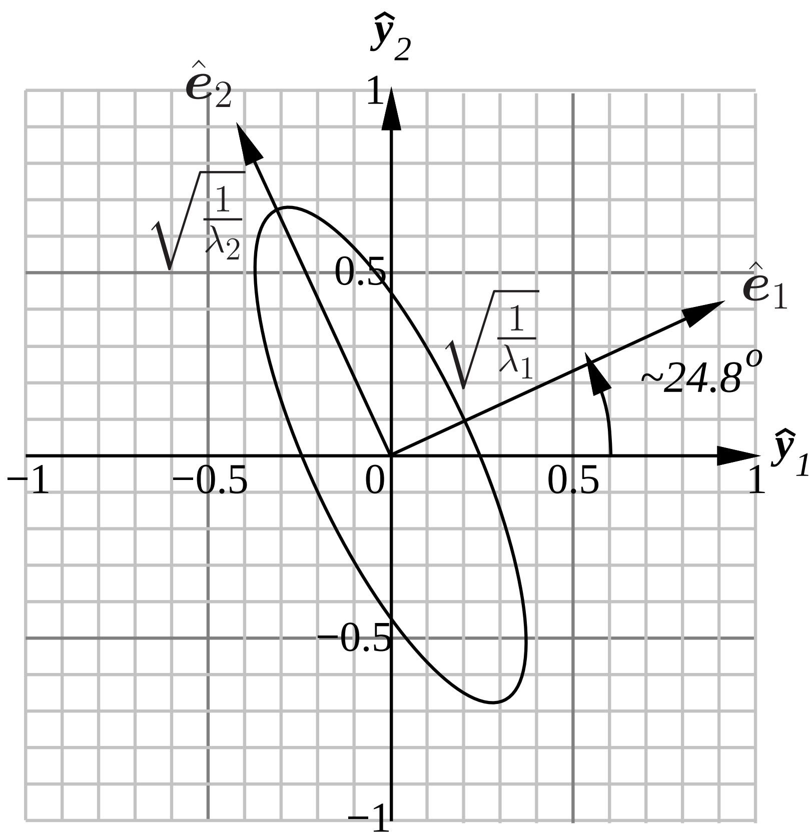
EXAMPLE: Ellipsoidal Sets from Quadratic Forms

$$\mathcal{E} = \{\ \boldsymbol{y} \mid \boldsymbol{y}^T \mathbf{M} \boldsymbol{y} \le 1\} \text{ in } \mathbb{R}^2$$

$$\boldsymbol{y}^T \begin{bmatrix} 17 & 7 \\ 7 & 5 \end{bmatrix} \boldsymbol{y} \le 1$$

$$\lambda_1 = 20.22 : \qquad \lambda_2 = 1.78 :$$

$$\hat{\boldsymbol{e}}_1 = \begin{bmatrix} 0.9085 \\ 0.4179 \end{bmatrix} \qquad \hat{\boldsymbol{e}}_2 = \begin{bmatrix} -0.4179 \\ 0.9085 \end{bmatrix}$$



Manipulator Velocity *Conditioning*

consider the manipulator Jacobian

$$\boldsymbol{v} = \mathbf{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and furthermore, consider mapping a unit hypersphere in the joint angle velocities through the Jacobian to Cartesian velocities

$$\|\dot{\boldsymbol{q}}\|^2 = \dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} = \dot{q}_0^2 + \dot{q}_1^2 + \ldots + \dot{q}_m^2 \leq 1,$$

$$\dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} = (\mathbf{J}^{-1}\boldsymbol{v})^T(\mathbf{J}^{-1}\boldsymbol{v}) = \boldsymbol{v}^T\left[(\mathbf{J}^{-1})^T\mathbf{J}^{-1}\right]\boldsymbol{v} = \boldsymbol{v}^T\left(\mathbf{J}\mathbf{J}^T\right)^{-1}\boldsymbol{v} \leq 1.$$

input hypersphere $\dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} \leq 1 \mapsto$ output hyperellipsoid that satisfies $\boldsymbol{v}^T\left(\mathbf{J}\mathbf{J}^T\right)^{-1}\boldsymbol{v} \leq 1$

quadratic form $\boldsymbol{v}^T[JJ^T]^{-1}\boldsymbol{v} \leq 1$ defines the "velocity conditioning ellipsoid" that reveals the directional sensitivity of the kinematic transformation.

Force Conditioning

the same analysis can be applied to the transformation of torques to Cartesian forces using the same manipulator Jacobian

$$\boldsymbol{\tau} = \mathbf{J}^T \boldsymbol{f}, \text{ so that}$$

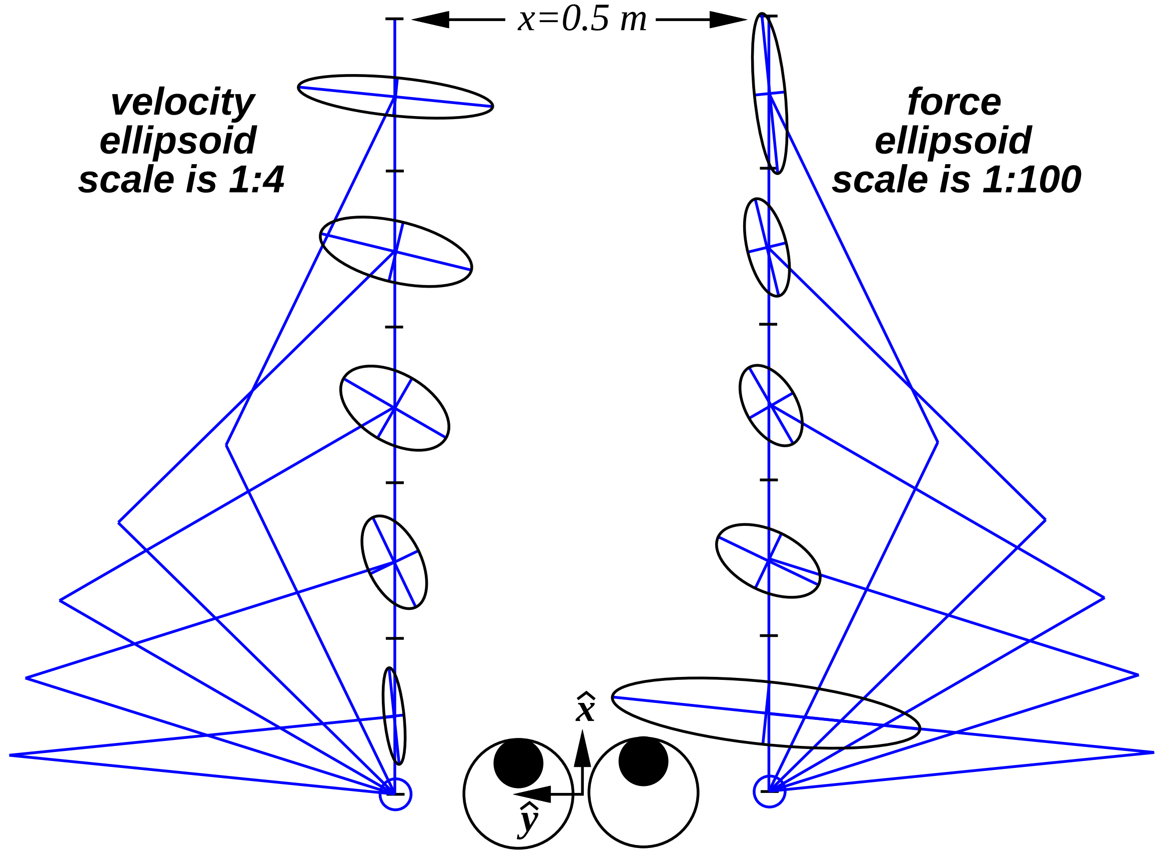
$$\boldsymbol{\tau}^T \boldsymbol{\tau} = (\mathbf{J}^T \boldsymbol{f})^T (\mathbf{J}^T \boldsymbol{f}) = \boldsymbol{f}^T (\mathbf{J}\mathbf{J}^T) \boldsymbol{f} \leq 1$$

the Cartesian force capacity of a particular posture in the manipulator is reflected by the ellipsoidal set

$$\{\boldsymbol{f} | \boldsymbol{f}^T (\mathbf{J}\mathbf{J}^T)\ \boldsymbol{f} \leq 1\}$$

- the eigenvectors of $(\mathbf{J}\mathbf{J}^T)$ and $(\mathbf{J}\mathbf{J}^T)^{-1}$ are identical
- eigenvalues of $(\mathbf{J}\mathbf{J}^T)^{-1}$ (velocity amplifier) are reciprocals of the eigenvalues of $(\mathbf{J}\mathbf{J}^T)$ (force amplifier).

Force Conditioning



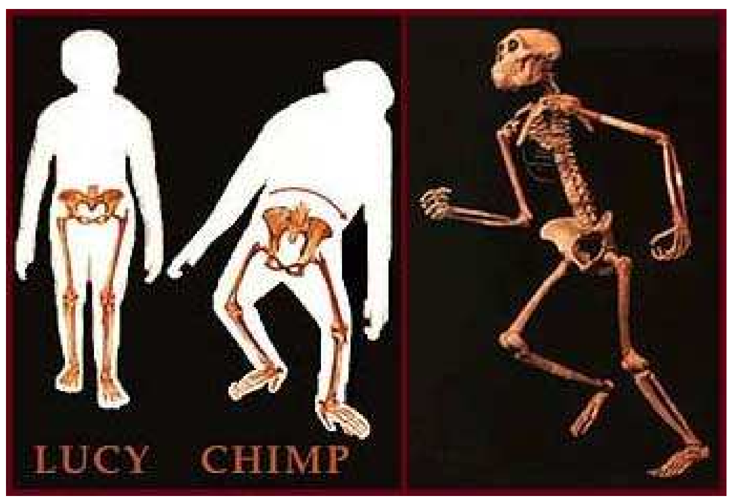
"...posture variation is a means through which motion and strength characteristics of the arm is made compatible with the task [**Chiu87**]."

The Kinematics of Bipedalism

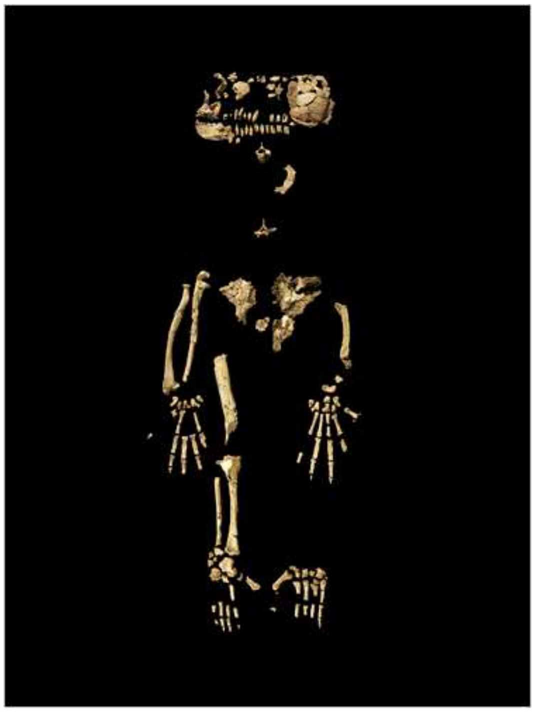
discovered in 1978 in Ethiopia by Mary Leakey, Australopithecus afarensis is classified as an ape, not a human. It is a **hominid**–an ape closely related to human beings in terms of overall body size, brain size and skull shape. A. afarensis lived 2.9-3.9 million years ago—Lucy was dated around 3.75 miliion years old.

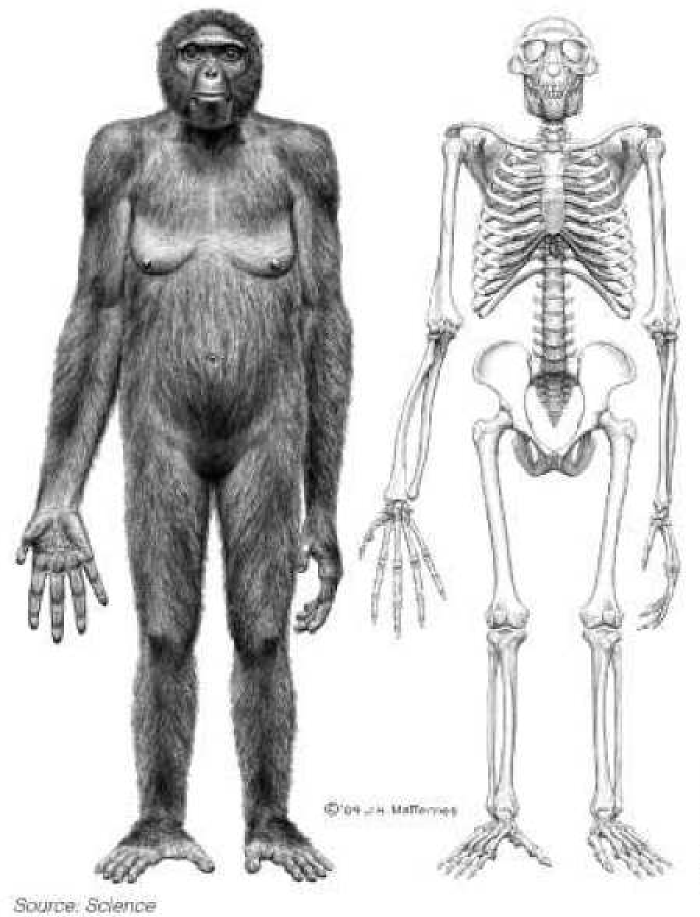
contemporary lower body
"chimp" hand
long arms, short legs
knee and pelvis imply:

- efficient climber, probably spent time in the trees, and
- was bipedal on the ground - walked much like we do.



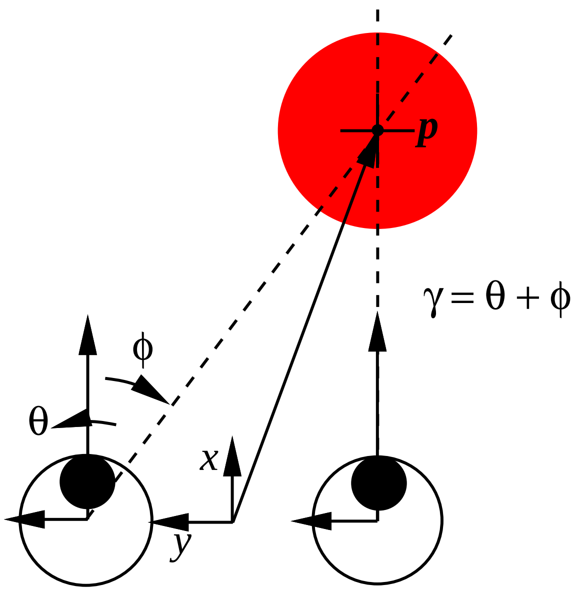
Ardipithecus ramidus - Ardi





- found in Ethiopia near where Lucy was found
- dated to 4.4 million years old, about 4 feet tall, 120 pounds
- long arms/hands, short legs, prehensile foot
- a climbing biped

Stereo Conditioning - Localizability



... from before,

$$p_x = 2d\frac{cos(\gamma_R)cos(\gamma_L)}{sin(\gamma_R - \gamma_L)}$$

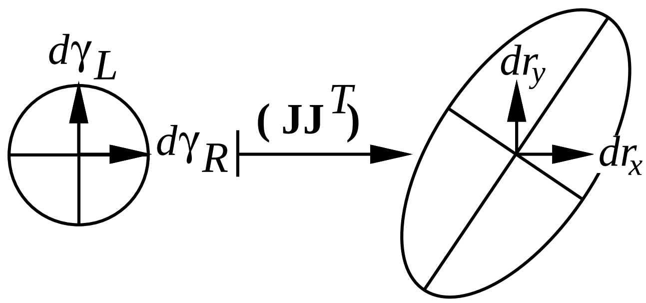
$$p_y = d + 2d\frac{cos(\gamma_R)sin(\gamma_L)}{sin(\gamma_R - \gamma_L)}.$$

$$\begin{bmatrix} dp_x \\ dp_y \end{bmatrix} = \begin{bmatrix} \frac{\partial p_x(\gamma_L,\gamma_R)}{\partial \gamma_L} & \frac{\partial p_x(\gamma_L,\gamma_R)}{\partial \gamma_R} \\ \frac{\partial p_y(\gamma_L,\gamma_R)}{\partial \gamma_L} & \frac{\partial p_y(\gamma_L,\gamma_R)}{\partial \gamma_R} \end{bmatrix} \begin{bmatrix} d\gamma_L \\ d\gamma_R \end{bmatrix}$$

$$= \frac{2d}{sin^2(\gamma_R - \gamma_L)} \begin{bmatrix} cos^2(\gamma_R) & -cos^2(\gamma_L) \\ sin(\gamma_R)cos(\gamma_R) & -sin(\gamma_L)cos(\gamma_L) \end{bmatrix} \begin{bmatrix} d\gamma_L \\ d\gamma_R \end{bmatrix}$$

relates velocities on the retina to velocities of the ball

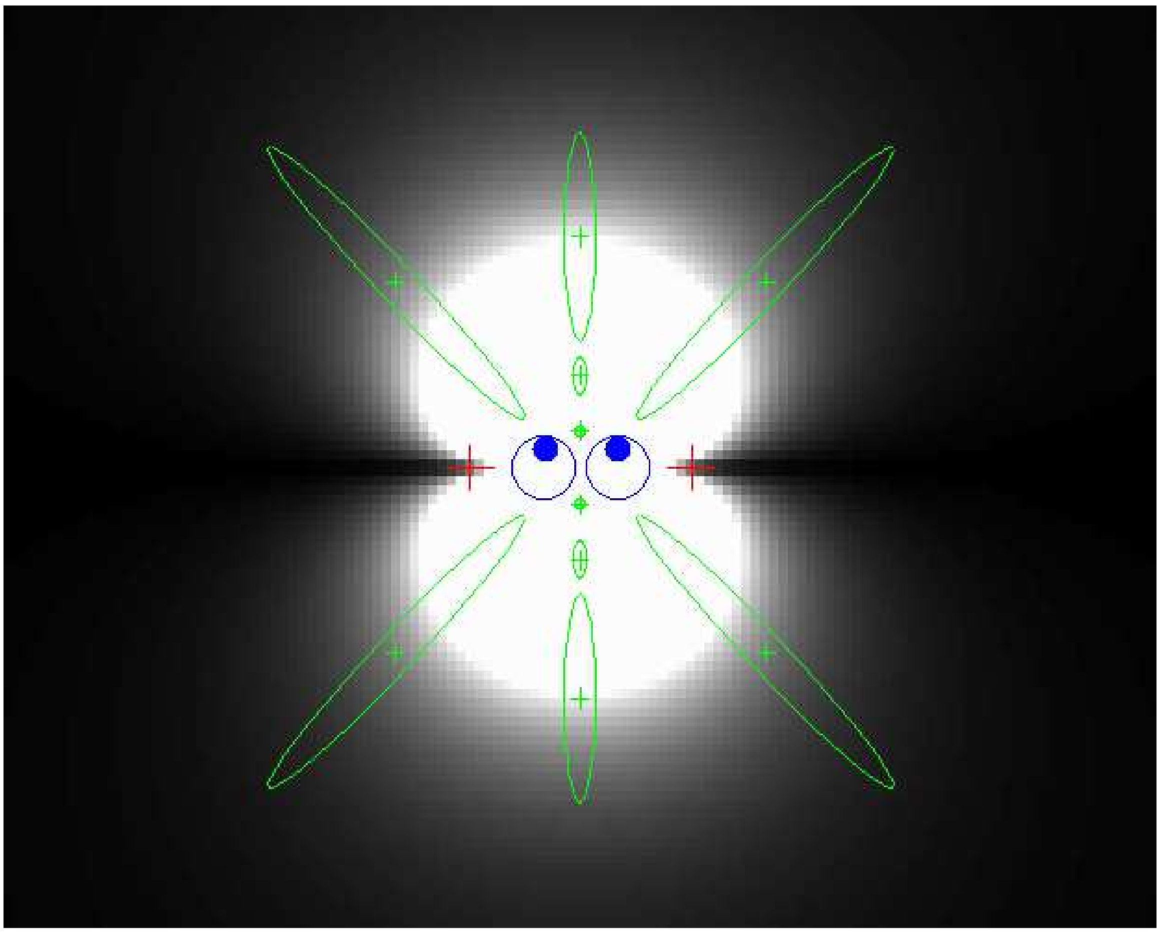
Localizability Ellipsoid



$$\begin{aligned} d\boldsymbol{\gamma}^T d\boldsymbol{\gamma} &= (\mathbf{J}^{-1} d\boldsymbol{r})^T (\mathbf{J}^{-1} d\boldsymbol{r}) \leq 1 \\ &= d\boldsymbol{r}^T (\mathbf{J}^{-1})^T \mathbf{J}^{-1} d\boldsymbol{r} \leq 1 \\ &= d\boldsymbol{r}^T (\mathbf{J}\mathbf{J}^T)^{-1} d\boldsymbol{r} \leq 1 \end{aligned}$$

if $d\boldsymbol{\gamma}$ represents the detection error on the retina, then the ellipsoidal set $\{\boldsymbol{r} | d\boldsymbol{r}^T (\mathbf{J}\mathbf{J}^T)^{-1} d\boldsymbol{r} \leq 1\}$ describes how the triangulation equations map retinal errors into Cartesian errors.

Localizability Ellipsoid



Summary: Kinematic Conditioning

manipulator conditioning

	amplification	precision
velocity	$\dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}} = \boldsymbol{v}^T [\mathbf{J}\mathbf{J}^T]^{-1} \ \boldsymbol{v} \leq 1$	$\boldsymbol{\epsilon}_{\dot{q}}^T \boldsymbol{\epsilon}_{\dot{q}} = \boldsymbol{\epsilon}_v^T [\mathbf{J}\mathbf{J}^T] \ \boldsymbol{\epsilon}_v \leq 1$
force	$\boldsymbol{\tau}^T \boldsymbol{\tau} = \boldsymbol{f}^T [\mathbf{J}\mathbf{J}^T] \boldsymbol{f} \leq 1$	$\boldsymbol{\epsilon}_{\tau}^T \boldsymbol{\epsilon}_{\tau} = \boldsymbol{\epsilon}_f^T [\mathbf{J}\mathbf{J}^T]^{-1} \boldsymbol{\epsilon}_f \leq 1$

visual acuity: $d\boldsymbol{\gamma}^T d\boldsymbol{\gamma} = d\boldsymbol{r}^T (\mathbf{J}\mathbf{J}^T)^{-1} d\boldsymbol{r} \leq 1$

- $\mathbf{J}\mathbf{J}^T$ is positive definite, symmetric, and square in the dimension of the output space.
- The principal axes of the conditioning ellipsoid are the eigenvectors of $\mathbf{M}$ in the quadratic form and the amplification in these directions are proportional to $\sqrt{1/\lambda}$.
- The conditioning ellipsoid represents configuration dependent anisotropy in a linear transform—principal axes are principal transformations of the governing Jacobian and describe amplification in a kinematic device.