Dynamics

The branch of physics that treats the action of force on bodies in motion or at rest; kinetics, kinematics, and statics, collectively. — Websters dictionary

Outline

• Conservation of Momentum
• Inertia Tensors - translation and rotation
• Dynamics
  – Newton/Euler Dynamics
  – State Space Form - computed torque equation
• Applications:
  – simulation;
  – control - feedforward compensation;
  – analysis - the acceleration ellipsoid.
Newton’s Laws

1. a particle will remain in a state of constant rectilinear motion unless acted on by an external force;

2. the time-rate-of-change in the momentum \((m \dot{v})\) of the particle is proportional to the externally applied forces, \(F = \frac{d}{dt}(m \dot{v})\); and

3. any force imposed on body \(A\) by body \(B\) is reciprocated by an equal and opposite reaction force on body \(B\) by body \(A\).

Conservation of Momentum

Linear:
\[
F = \frac{d}{dt} [m \dot{x}] = m \ddot{x}
\]
\[
[N] = \left[ \frac{kg \ m}{sec^2} \right]
\]

Angular:
\[
\tau = \frac{d}{dt} \left[ J \dot{\theta} \right] = J \ddot{\theta}
\]
\[
[Nm] = \left[ \frac{kg \ m^2}{sec^2} \right]
\]

\(J\) is called the mass moment of inertia
Conservation of Momentum

To generate an angular acceleration about frame $O$, a torque is applied around the $\hat{z}$ axis

\[ \boldsymbol{\tau}_k = \boldsymbol{r} \times \boldsymbol{f} = r_k \hat{r} \times \frac{d}{dt} (m_k \boldsymbol{v}_k) \]

\[ = m_k r_k \left[ \hat{r} \times \frac{d}{dt} (\boldsymbol{v}_k) \right] \]

the velocity of $m_k$ due to $\omega_O$ is

\[ \boldsymbol{v}_k = (\omega \hat{z} \times r_k \hat{r}) = (r_k \omega) \hat{t}, \]

so that

\[ \boldsymbol{\tau}_k = (m_k r_k^2) \dot{\omega} \hat{z} = J_k \dot{\omega} \hat{z} \]

where, $J_k$ is the mass moment of inertia of particle $m_k$ around the $\hat{z}$ axis located at frame $O$

\[ \boldsymbol{\tau} = \left( \sum_k m_k r_k^2 \right) \dot{\omega} = J \dot{\omega}. \]
Conservation of Momentum

\[ J \text{ [kg} \cdot \text{m}^2\text{]} \] is the scalar moment of inertia

In the rotating lamina, the counterpart of linear momentum \((p = mv)\) is angular momentum \(L = J\omega\),

so that Euler’s equation can be written in the same form as Newton’s second law

\[ \tau = \frac{d}{dt}[J\omega] = J\ddot{\theta} \]

...rotating bodies conserve angular momentum (and remain in a constant state of angular velocity) unless acted upon by an external torque...

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Inertia Tensor

\[ I = \begin{bmatrix}
  I_{xx} & -I_{xy} & -I_{xz} \\
  -I_{yx} & I_{yy} & -I_{yz} \\
  -I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix} \]

**Mass Moments of Inertia**

- \( I_{xx} = \int \int \int (y^2 + z^2) \rho \, dv \)
- \( I_{yy} = \int \int \int (x^2 + z^2) \rho \, dv \)
- \( I_{zz} = \int \int \int (x^2 + y^2) \rho \, dv \)

**Mass Products of Inertia**

- \( I_{xy} = \int \int \int xy \rho \, dv \)
- \( I_{xz} = \int \int \int xz \rho \, dv \)
- \( I_{yz} = \int \int \int yz \rho \, dv \)
EXAMPLE: Inertia Tensor of an Eccentric Rectangular Prism

\[ I_{xx} = \int_0^h \int_0^l \int_0^w (y^2 + z^2) \rho dxdydz \]
\[ = \int_0^h \int_0^l (y^2 + z^2) w \rho dydz \]
\[ = \int_0^h \left[ \left( \frac{y^3}{3} + z^2 y \right) \right]_0^l w \rho dz \]
\[ = \int_0^h \left( \frac{l^3}{3} + z^2 l \right) w \rho dz \]
\[ = \left. \left( \frac{l^3 z}{3} + \frac{l z^3}{3} \right) \right|_0^h (w \rho) \]
\[ = \left( \frac{l^3 h}{3} + \frac{l h^3}{3} \right) w \rho \]

or, since the mass of the rectangle \( m = (wlh) \rho \),

\[ I_{xx} = \frac{m}{3} (l^2 + h^2). \]
EXAMPLE: Inertia Tensor of an Eccentric Rectangular Prism

...completing the other moments and products of inertia yields:

\[ A I = \begin{bmatrix} \frac{m}{3}(l^2 + h^2) & \frac{m}{4}wl & \frac{m}{4}hw \\ \frac{m}{4}wl & \frac{m}{3}(w^2 + h^2) & \frac{m}{4}hl \\ \frac{m}{4}hw & \frac{m}{4}hl & \frac{m}{3}(l^2 + w^2) \end{bmatrix} \]
Parallel Axis Theorem -
Translating the Inertia Tensor

the moments of inertia look like:

\[ A I_{zz} = CM I_{zz} + m(r_x^2 + r_y^2), \]

and the products of inertia are:

\[ A I_{xy} = CM I_{xy} + m(r_x r_y). \]
EXAMPLE: The Symmetric Rectangular Prism

\[ CM I_{zz} = A I_{zz} - m(r_x^2 + r_y^2) \]
\[ = \frac{m}{3}(l^2 + w^2) - \frac{m}{4}(l^2 + w^2) \]
\[ = \frac{m}{12}(l^2 + w^2) \]

\[ CM I_{xy} = A I_{xy} - m(r_x r_y) \]
\[ = \frac{m}{4}(wl) - \frac{m}{4}(wl) = 0. \]

Moving the axes of rotation to the center of mass results in a diagonalized inertia tensor

\[
CM I = \frac{m}{12} \begin{bmatrix}
(l^2 + h^2) & 0 & 0 \\
0 & (w^2 + h^2) & 0 \\
0 & 0 & (l^2 + w^2)
\end{bmatrix}
\]

diagonal terms are smaller and the off-diagonals are 0
Rotating the Inertia Tensor

angular momentum \( \mathbf{L}_0 = \mathbf{I}_0\mathbf{\omega} \) about frame 0 in a vector quantity that is conserved.

we can express it relative to frame 1 as

\[
\mathbf{L}_1 = {}_1\mathbf{R}_0\mathbf{L}_0
\]

or

\[
\mathbf{I}_1\mathbf{\omega}_1 = {}_1\mathbf{R}_0(\mathbf{I}_0\mathbf{\omega}_0)
\]

\[
= {}_1\mathbf{R}_0\mathbf{I}_0[{}_1\mathbf{R}_0^T {}_1\mathbf{R}_0]\mathbf{\omega}_0 = {}_1\mathbf{R}_0\mathbf{I}_0 {}_1\mathbf{R}_0^T \mathbf{\omega}_1
\]

and therefore,

\[
\mathbf{I}_1 = {}_1\mathbf{R}_0 \mathbf{I}_0 {}_1\mathbf{R}_0^T.
\]
Rotating Coordinate Systems

**Definition (Inertial Frame)**
the frame where the absolute state of motion is completely known

Let frame \( A \) be an inertial frame. Frame \( B \) has an absolute velocity, \( \omega_B \) (written in terms of frame \( B \) coordinates).

\[
\begin{align*}
\mathbf{r}_A(t) &= A R_B(t) \mathbf{r}_B(t) \\
\dot{\mathbf{r}}_A(t) &= A R_B(t) \frac{d}{dt} [\mathbf{r}_B(t)] + \frac{d}{dt} [ A R_B(t) ] \mathbf{r}_B(t)
\end{align*}
\]

To evaluate the second term on the right, consider how the \( \hat{x}, \hat{y}, \) and \( \hat{z} \), basis vectors for frame \( B \) change by virtue of \( \omega_B \).

\[
\begin{align*}
\dot{\hat{x}} &= +\omega_z \hat{y} - \omega_y \hat{z} \\
\dot{\hat{y}} &= -\omega_z \hat{x} + \omega_x \hat{z} \\
\dot{\hat{z}} &= +\omega_y \hat{x} - \omega_x \hat{y}
\end{align*}
\]

So

\[
\frac{d}{dt} [ A R_B(t) ] \mathbf{r}_B(t) = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \omega \times \mathbf{r}
\]
Rotating Coordinate Systems

Therefore,

\[ \dot{r}_A(t) = A R_B(t) \frac{d}{dt}[r_B(t)] + \frac{d}{dt}[A R_B(t)] r_B(t) \]
\[ = A R_B [\dot{r}_B + (\omega_B \times r_B)] \]

and, in fact, all vector quantities expressed in local frames that are moving relative to an inertial frame are differentiated in this way

\[ \frac{d}{dt}[A R_B(t)(\cdot)_B] = A R_B \left[ \frac{d}{dt}(\cdot)_B + (\omega_B \times (\cdot)_B) \right] \]

gives rise to the notorious Coriolis and centripetal forces!
large scale atmospheric flows converge at low pressure regions. A nonrotating planet, this would result in flow lines directed radially inward. but the earth rotates...

consider a stationary inertial frame $A$ and a rotating frame $B$ attached to the earth

\[
\mathbf{v}_A = A R_B(t) \mathbf{v}_B
\]

\[
\dot{\mathbf{v}}_A = A R_B [\dot{\mathbf{v}}_B + (\omega \times \mathbf{v}_B)]
\]

so that to an observer that travels with frame $B$:

\[
\dot{\mathbf{v}}_B = B R_A [\dot{\mathbf{v}}_A] - (\omega \times \mathbf{v}_B)
\]

a convergent flow and a rotating system, therefore, leads to a counterclockwise flow in the northern hemisphere and a clockwise rotation in the southern hemisphere.
Newton/Euler Method

The recursive equations for these iterations are derived in Appendix B of the book.
Recursive Newton-Euler Equations

Outward Iterations

Angular Velocity: \( \omega \)
- REVOLUTE: \( ^{i+1} \omega_{i+1} = ^{i+1} R_i \dot{\omega}_i + \dot{\theta}_{i+1} \hat{z}_{i+1} \)
- PRISMATIC: \( ^{i+1} \omega_{i+1} = ^{i+1} R_i \dot{\omega}_i \)

Angular Acceleration: \( \dot{\omega} \)
- REVOLUTE: \( ^{i+1} \dot{\omega}_{i+1} = ^{i+1} R_i \dot{\omega}_i + ( ^{i+1} R_i \omega_i \times \dot{\theta}_{i+1} \hat{z}_{i+1} ) + \ddot{\theta}_{i+1} \hat{z}_{i+1} \)
- PRISMATIC: \( ^{i+1} \dot{\omega}_{i+1} = ^{i+1} R_i \dot{\omega}_i \)

Linear Acceleration: \( \dot{v} \)
- REVOLUTE: \( ^{i+1} \dot{v}_{i+1} = ^{i+1} R_i \left[ ^i \dot{v}_i + ( ^i \omega_i \times ^i p_{i+1} ) + ( ^i \omega_i \times ( ^i \hat{p}_{i+1} \times ^i \dot{\omega}_i ) ) \right] \)
- PRISMATIC: \( ^{i+1} \dot{v}_{i+1} = ^{i+1} R_i \left[ ^i \dot{v}_i + \dot{\hat{d}}_i \hat{x}_i + 2( ^i \omega_i \times \dot{d}_i \hat{x}_i ) + ( ^i \dot{\omega}_i \times ( ^i \hat{d}_i \hat{x}_i ) ) + ( ^i \omega_i \times ^i \omega_i \times ^i \hat{d}_i \hat{x}_i ) \right] \)

Linear Acceleration (center of mass): \( \dot{v}_{cm} \)
- \( ^{i+1} \dot{v}_{cm,(i+1)} = ( ^{i+1} \dot{\omega}_{i+1} \times ^{i+1} p_{cm} ) + ( ^{i+1} \omega_{i+1} \times ^{i+1} \dot{p}_{cm} ) + ^{i+1} \dot{v}_{i+1} \)

Net Force: \( F \)
- \( ^{i+1} F_{i+1} = m_{i+1} ^{i+1} \dot{v}_{cm} \)

Net Moment: \( N \)
- \( ^{i+1} N_{i+1} = I_{i+1} ^{i+1} \dot{\omega}_{i+1} + ( ^{i+1} \omega_{i+1} \times I_{i+1} ^{i+1} \omega_{i+1} ) \)

Inward Iterations

Inter-Link Forces:
- \( ^i f_i = ^i F_i + ^i R_{i+1} ^{i+1} f_{i+1} \)

Inter-Link Moments:
- \( ^i \eta_i = ^i N_i + ^i R_{i+1} ^{i+1} \eta_{i+1} + ( ^i p_{cm} \times ^i F_i ) + ( ^i p_{i+1} \times ^i R_{i+1} ^{i+1} f_{i+1} ) \)
The Computed Torque Equation

State Space Form

\[ \tau = M(\theta)\ddot{\theta} + V(\theta \dot{\theta}) + G(\theta) + F \]

external forces/torques:

- external forces
- friction
  - viscous \( \tau = -v\dot{\theta} \)
  - coulomb \( \tau = -c(\text{sgn}(\dot{\theta})) \)
  - hybrid
EXAMPLE: Dynamic Model of Roger’s Eye

\[ \sum \tau = \frac{d}{dt} (J \dot{\theta}) \]

\[ \tau_m + mgl\sin(\theta) = (ml^2) \ddot{\theta} \]

or

\[ \tau_m = M\ddot{\theta} + G, \]

generalized inertia

\[ M = ml^2 \text{ (a scalar)}; \]

Coriolis and centripetal forces

\[ \mathbf{V}(\theta, \dot{\theta}) \text{ do not exist; and} \]

Gravitational loads

\[ G = -mgl\sin(\theta) \]
EXAMPLE: Dynamic Model of Roger’s Arm

\[ M(\theta) = \begin{bmatrix}
    m_2 l_2^2 + 2m_2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\
    m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2
\end{bmatrix} \]

\[ V(\theta, \dot{\theta}) = \begin{bmatrix}
    -m_2 l_1 l_2 s_2 (\dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2) \\
    m_2 l_1 l_2 s_2 \dot{\theta}_1^2
\end{bmatrix} \quad Nm \]

\[ G(\theta) = \begin{bmatrix}
    -(m_1 + m_2) l_1 s_1 g - m_2 l_2 s_{12} g \\
    -m_2 l_2 s_{12} g
\end{bmatrix} \quad Nm \]
EXAMPLE: Roger’s Whole-Body Dynamics

Roger’s whole-body dynamics can also be written in the standardized form of the computed torque equation.

$$\tilde{\tau} = M \ddot{q} + V(q, \dot{q}) + F$$

where $\tilde{\tau} \in \mathbb{R}^8$ is the vector of forces and torques causing acceleration in the degrees of freedom $q \in \mathbb{R}^8$ of the robot.
Simulation

\[ \ddot{\theta} = M^{-1}(\theta) \left[ \tau - V(\theta \dot{\theta}) - G(\theta) - F \right] \]

initial conditions:

\[ \theta(0) = \theta_0 \quad \dot{\theta}(0) = \ddot{\theta}(0) = 0 \]

numerical integration:

\[ \ddot{\theta}(t) = M^{-1}[\tau - V - G - F] \]

\[ \dot{\theta}(t + \Delta t) = \dot{\theta}(t) + \ddot{\theta}(t)\Delta t \]

\[ \theta(t + \Delta t) = \theta(t) + \dot{\theta}(t)\Delta t + \frac{1}{2} \ddot{\theta}(t)\Delta t^2 \]
Feedforward Dynamic Compensators

linearized and decoupled
Generalized Inertia Ellipsoid

computed torque equation:

$$\tau = M\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta)$$

if we assume that $\dot{\theta} \approx 0$, and we ignore gravity

$$\tau = M\ddot{\theta}$$

$$\|\ddot{\theta}\| \leq 1$$

relative inertia—torque required to create a unit acceleration defined by the eigenvalues and eigenvectors of $MM^T$
Acceleration Polytope

Gravity, actuator performance, and the current state of motion influences the ability of a manipulator to generate accelerations differentiating $\ddot{r} = J\dot{q},$

\[
\ddot{r} = J(q)\ddot{q} + \dot{J}(q, \dot{q})\dot{q}
\]

\[
= J\left[ M^{-1}(\tau - V - G) \right] + \dot{J}\dot{q}
\]

\[
= JM^{-1}\tau + \dot{v}_{vel} + \dot{v}_{grav},
\]

\[
\dot{v}_{vel} = -JM^{-1}V + \dot{J}\dot{q}, \quad \text{and}
\]

\[
\dot{v}_{grav} = -JM^{-1}G.
\]

$\tilde{\tau} = L^{-1}\tau \quad L = \text{diag}(\tau_{1}^{\text{limit}}, \ldots, \tau_{n}^{\text{limit}})$

Admissible torques constitute a unit hypercube $\|\tilde{\tau}\|_{\infty} \leq 1$

\[
\ddot{r} = JM^{-1}L\tilde{\tau} + \dot{v}_{vel} + \dot{v}_{grav}
\]

\[
= JM^{-1}L\tilde{\tau} + \dot{v}_{bias}.
\]

Maps the $n$-dimensional hypercube $\|\tilde{\tau}\|_{\infty} \leq 1$ to the $m$-dimensional acceleration polytope.
\[ \tilde{\tau}^T \tilde{\tau} = (\ddot{r} - \dot{v}_{bias})^T \left( [JM^{-1}L]^{-1} \right)^T \left( [JM^{-1}L]^{-1} \right) (\ddot{r} - \dot{v}_{bias}) \leq 1 \]  

\( M \) and \( L \) are symmetric:

\[ A^{-T} = (A^{-1})^T, \quad A^{-2} = A^{-1}A^{-1}, \text{ and for symmetric matrices,} \quad A^T = A. \]

\[ (\ddot{r} - \dot{v}_{bias})^T \left[ J^{-T}ML^{-2}MJ^{-1} \right] (\ddot{r} - \dot{v}_{bias}) \leq 1, \]

so that

**dynamic manipulability ellipsoid**

\[ (\ddot{r} - \dot{v}_{bias})(\ddot{r} - \dot{v}_{bias})^T \in [JM^{-T}L^2M^{-1}J^T] \]

**dynamic-manipulability measure**

\[ \kappa_d(q, \dot{q}) = \sqrt{\det [J(M^TM)^{-1}J^T]} \]
Conditioning Acceleration

\[ m_1 = m_2 = 0.2 \text{ kg}, \quad l_1 = l_2 = 0.25 \text{ m}, \quad \tau^T \tau \leq 0.005 \text{ N}^2\text{m}^2. \]

black ellipsoids - unbiased dynamic manipulability
gravity biased dynamic manipulability
normalized acceleration polytope with gravity bias